Pontificia Universidad Católica de Chile
Facultad de Física
Departamento de Física

Quantisation of cosmological perturbations in the Eddington-Born-Infeld theory

by

Macarena Alejandra Lagos Urbina

Thesis presented to the Physics Faculty
at Pontificia Universidad Católica de Chile for
the Master's degree in Physics

Supervisor : Dr. Máximo Bañados (PUC, Chile)
Correctors : Dr. Jorge Alfaro (PUC, Chile)
Dr. Andreas Reisenegger (PUC, Chile)

June 2013
Santiago – Chile

Acknowledgments

I thank my supervisor, Professor Máximo Bañados, for giving me the opportunity to work in this interesting topic. I am also very grateful for the time he has dedicated to this work, his attention, advice and good ideas, which have been an invaluable guide for me during my Master's studies. I finally thank him for his concern for my future career plans.

I also thank to CONICYT for funding my master's studies.

Lastly, I would like to thank Gonzalo Díaz, Mabel Urbina and Claudia Lagos, for their love and constant support.

Summary

In this thesis we studied the origin of cosmological perturbations in the early Universe, which were seeds of temperature anisotropies and structures observed today, such as galaxies, clusters, etc. Initially, we reviewed the well-known theory of inflation in General Relativity, including the motivations, main assumptions, gauge freedoms, quantum theory and predictions. We showed that inflation predicts a nearly scale-free power spectrum for primordial scalar quantum fluctuations, which grow in time and generate inhomogeneities today. This result is in agreement with observations (see Komatsu et al. (2011)). It also predicts a nearly scale-free power spectrum for tensor perturbations. In addition, we studied this issue in a modified gravity called Eddington-Born-Infeld theory (Bañados and Ferreira (2010)), which introduces significant modifications to General Relativity in regions with high curvature, e.g. during the early Universe. We discussed the motivations, gauge freedoms, quantum theory and predictions. We found a scale-free power spectrum for scalar and tensor primordial quantum perturbations, although some numerical estimations do not fit the experimental values.

Contents

Chapter 1

Introduction

The Universe is very homogeneous and isotropic on large scales ($\sim 100\,\mathrm{Mpc}$). Observations of the Cosmic Microwave Background (CMB) by WMAP[1] and matter distribution by SDSS[2] evidence the isotropy. Since we assume that we take no special place in the Universe, we can infer homogeneity. However, if we look at the sky today, we can see structures, such as galaxies, clusters and superclusters. These inhomogeneities have also been observed in the CMB. The temperature of the CMB is $T_0 \approx 2.7255\,\mathrm{K}$, but small temperature anisotropies of the order of $\frac{\delta T}{T_0} \sim 10^{-5}$ appear. The description of the origin and evolution of these inhomogeneities is an open problem in cosmology.

In order to explain these observed inhomogeneities, it is assumed that there was an early stage where the Universe was nearly homogeneous and isotropic. At this stage, some primordial quantum fluctuations are created, which grow in time, and eventually can generate the structures observed today due to gravitational instability. This is the motivation to study early linear perturbations in a cosmological background. Then, mathematically, the problem of describing the growth of initial small perturbations is reduced to finding a quantum solution for the fluctuations during the early Universe. This process is full of subtleties related to the gauge symmetry of General Relativity, physical variables and quantum vacuum choices, which will be explained in this thesis.

The most accepted theory for describing these primordial fluctuations is the inflation

[1] Wilkinson Microwave Anisotropy Probe.

[2] Sloan Digital Sky Survey.

theory for General Relativity, originally proposed by Guth (1981). It initially appears to solve some shortcomings of the Big Bang model, such as the horizon and flatness problems. Inflation is an early stage of the Universe where the energy of a scalar field dominates. During this era, the scale factor grows nearly exponentially, so the expansion of the Universe is accelerating. Inflation assumes that linear quantum fluctuations are in their ground state in the past, and it makes a prediction for them: a nearly scale-invariant power spectrum. It also assumes Gaussian and adiabatic fluctuations. It is possible to compare these predictions with observations. In fact, this result is in agreement with observations of the CMB anisotropies and matter distribution (Komatsu et al. (2011)). Even though the inflation theory has been very successful due to its important predictions, it has also been criticised for its assumption of a fundamental unknown scalar field, with very peculiar characteristics.

One possible alternative to the inflation theory is given by the Eddington-Born-Infeld (EBI) gravity (Bañados and Ferreira (2010)). This is a modified gravity, which was originally proposed with the intention of eliminating divergences present in General Relativity. As a result, it introduces significant modifications during the early Universe, including the elimination of the Big Bang. Also, it predicts a period of accelerated expansion during the radiation-dominated era, which avoids the horizon and flatness problems. These results suggest that this theory could be an alternative to inflation, and then motivate the study of the origin and evolution of primordial quantum fluctuations in the EBI theory.

A different approach to explaining the origin of fluctuations in General Relativity is proposed by Hollands and Wald (2008). Here, a scale-invariant power spectrum for primordial fluctuations is obtained without introducing a fundamental scalar field. This approach is based on the assumption that semiclassical physics applies to scales greater than a fundamental scale l_0, so perturbations can be considered as born in their ground state when the physical wavelength λ_{ph} of a perturbation satisfies $\lambda_{ph} = l_0$.

In this thesis we will explain the theory of inflation and study inhomogeneities in the EBI theory by making use of the ideas proposed by Hollands and Wald. The text is organised as follows:

In chapter 2 we review the theory of inflation along with its key predictions. We first discuss the main picture and shortcomings of the Big Bang model. We also introduce the linear perturbation theory in a homogeneous and isotropic background Universe. We show the standard classification of metric perturbations. After that, we motivate the theory of inflation by explaining the solution it gives to some shortcomings of the Big Bang theory. We show the most common implementation of inflation: a single scalar field minimally coupled to gravity, with a slow-roll potential. Then we discuss the importance of the gauge degrees of freedom in the quantum theory of inflation, along with the method to fix them correctly. Finally, we focus on the scalar perturbations in inflation and we quantise them. We calculate the prediction for the power spectrum of the only scalar physical degree of freedom present in this theory, and mention the result in the case of tensor perturbations.

Chapter 3 follows the same basic structure as chapter 2, now for the Eddington-Born-Infeld theory (Bañados and Ferreira (2010)). First, we introduce this gravitational theory by giving a motivation and showing the form of the action. After that, we show the evolution of the scale factor and its cosmological consequences in the homogeneous and isotropic Universe. With the same motivation of chapter 2, we consider the linear theory of perturbations, focusing on scalar and tensor fields. Then we revisit the gauge freedoms and show a specific gauge choice. After that, we explicitly calculate a second-order EBI action coupled to a perfect fluid. Finally, we quantise the physical scalar and tensor degrees of freedom in this theory and calculate their spectra, following the approach of Hollands and Wald (2008).

Chapter 2

Cosmological quantum perturbations in General Relativity

In this chapter we will review the theory of inflation for primordial linear quantum perturbations. First, we will show the Einstein equations in a homogeneous and isotropic background coupled to a perfect fluid. This model corresponds to the Big Bang theory. Next, with the motivation of studying the origin of inhomogeneities, we will introduce the linear theory of perturbations. We will state the standard classification for general metric perturbations explicitly. Then we will introduce the inflation theory, including the problems it solves and the single scalar field model. We will then discuss some problems related to the gauge invariance of this theory, show the transformations of the perturbation fields and explain how to make a gauge choice correctly. Finally, we will review the quantum cosmological theory in inflation. We will focus on scalar perturbation fields, and calculate a second-order action for them, which will be quantised in order to find the power spectrum for primordial fluctuations. We will also mention briefly the result for tensor perturbations. Throughout this chapter we will use $8\pi G = 1$, $c = 1$, $\hbar = 1$ and signature (+,-,-,-).

2.1 Cosmological background and perturbations

In this section we review the main equations of the Big Bang model. Then, we study the cosmological theory of linear perturbations. The latter is needed to explain the origin of observed structures such as planets, stars, galaxies, etc.

2.1.1 Background: homogeneous and isotropic Universe

Cosmology describes the global structure and evolution of the Universe. The standard cosmology is based upon a maximally spatially symmetric Universe: homogeneous and isotropic. This assumption embodies the observational fact that the Universe is nearly homogeneous and isotropic on large scales. Therefore, as an approximation, the metric line element for the spacetime of the Universe is:

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{x} \cdot d\vec{x}), \tag{2.1}$$

where η is the conformal time and $a(\eta)$ is known as the scale factor. Here, we have assumed a spatially flat metric, i.e. a Universe with an Euclidean spatial geometry. This assumption is supported by observations (see Komatsu et al. (2011)), which show a nearly spatially flat Universe.

The contribution of matter and radiation to the Universe is well described for a perfect hydrodynamic approximation. The stress-energy tensor is then:

$$T^{\mu}{}_{\nu} = (p + \rho)u^{\mu}u_{\nu} - p\delta^{\mu}{}_{\nu}, \tag{2.2}$$

where $p = p(\eta)$ is the pressure of this fluid, $\rho = \rho(\eta)$ the rest energy density and $u^{\mu} = (1/a, 0, 0, 0)$ its isotropic 4-velocity.

Usually, a simple relation for the state equation for the perfect fluid is used:

$$p = \omega\rho, \tag{2.3}$$

where ω is constant. In order to study the evolution of the scale factor, in the Big Bang theory, there are 3 types of matter considered:

- Radiation (or relativistic matter): $\omega = 1/3$,
- Dust (or non-relativistic matter): $\omega = 0$,
- Dark energy (or vacuum energy): $\omega = -1$.

The equations of motion governing the evolution of the scale factor are the Einstein equations and the continuity equation for matter:

$$G^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}, \tag{2.4}$$

$$T^{\mu}{}_{\nu;\mu} = 0, \tag{2.5}$$

which, considering (2.1) and (2.2), are explicitly:

$$\mathcal{H}^2 = \frac{1}{3}\rho a^2, \tag{2.6}$$

$$a'' = \frac{1}{6}(\rho - 3p)a^3, \tag{2.7}$$

$$\rho' = -3\mathcal{H}(\rho + p), \tag{2.8}$$

where $' \equiv d/d\eta$, and $\mathcal{H} \equiv a'/a$ is the comoving parameter. Equation (2.6) is known as the Friedmann equation.

All the equations shown in this subsection give the basis of the Big Bang model. This model makes accurate and testable hypotheses in four key areas: expansion of the Universe, origin of the cosmic microwave background, nucleosynthesis of light elements, formation of galaxies and large-scale structures. The remarkable agreement with the observational data gives considerable confidence in the model. However, this model does not explain some features (Garcia-Bellido (2000)), such as flatness, large scale homogeneity and isotropy. These are severe shortcomings in the predictive power of the Big Bang model, which are the motivation for the inflation theory. It also lacks an explanation for the origin of inhomogeneities such as stars and galaxies. This problem is addressed by the theory of perturbations.

2.1.2 Theory of linear perturbations

Cosmological perturbation theory is an important area of study because it allows us to explain the fact that our Universe is not perfectly homogeneous and isotropic, as the Big Bang theory describes it. Our Universe has inhomogeneities, such as galaxies, clusters and CMB anisotropies.

It is natural to suppose that small deviations from homogeneity and isotropy were generated during the early Universe due to quantum physics. Since gravitation is an attractive force, these small perturbations would grow in time, producing structures through the mechanism of gravitational collapse. So, in order to get a more precise description of our Universe, it is important to study the origin and evolution of these small primordial perturbations. We do that using the linear perturbation theory.

Let us consider linear perturbations of the spacetime metric and matter on a given background. In general, we can split the perturbed metric into two parts:

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu}, \quad \text{such that} \quad |\delta g_{\mu\nu}| \ll |{}^{(0)}g_{\mu\nu}|, \tag{2.9}$$

where ${}^{(0)}g_{\mu\nu}$ corresponds to a background metric and $\delta g_{\mu\nu}$ to the first order metric corrections. This second term describes how the real spacetime deviates from the idealized background. Matter is also perturbed:

$$T^{\mu}{}_{\nu} = {}^{(0)}T^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}, \quad \text{such that} \quad |\delta T^{\mu}{}_{\nu}| \ll |{}^{(0)}T^{\mu}{}_{\nu}|, \tag{2.10}$$

where ${}^{(0)}T^{\mu}{}_{\nu}$ represents a background stress-energy tensor and $\delta T^{\mu}{}_{\nu}$ its perturbation. The background terms and first order terms satisfy the standard equations (2.4) and (2.5).

Notice that all perturbations are initially arbitrary, so even if the background is homogeneous and isotropic, in general perturbations are not.

Next, we show the standard way of writing metric perturbations.

2.1.3 Classification of metric perturbations

In the following discussion, we will consider first order perturbations in the early stages of the idealized homogeneous and isotropic background Universe described previously. Metric perturbations are categorised into three different types: scalar, vectorial and tensorial. This classification refers to the way perturbation fields transform under spatial coordinate transformations. Scalar metric perturbations are coupled to matter perturbations, so they are responsible of the formation of the observed structures today. Tensor perturbations give origin to primordial gravitational waves. For a perfect fluid, vector perturbations decay in an expanding Universe, so they are not relevant today (see Baccigalupi (2012)).

Scalar Perturbations

The most general way to construct metric perturbations to the background 2.1 with scalars, is using 4 scalar fields ϕ, ψ, E, B:

$$\delta^{(s)} g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}) \end{pmatrix}. \tag{2.11}$$

These ordinary partial derivatives are in general covariant derivatives with respect to a spatial background metric (see Mukanov, Feldman and Brandenberger (1992)).

Vector Perturbations

Vector perturbations can be constructed by using two vectors in 3 dimensions: S_i and F_i. These vectors satisfy:

$$S_i{}^{,i} = F_i{}^{,i} = 0. \tag{2.12}$$

In this way, these two vectors have no scalar part. Here we shift from upper to lower three-space indices and vice versa by using the metric δ_{ij} and its inverse δ^{ij}. So, a general vector metric perturbation is the following:

$$\delta^{(v)} g_{\mu\nu} = -a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_i & F_{i,j} + F_{j,i} \end{pmatrix}. \tag{2.13}$$

Tensor perturbations

Tensor perturbations can be constructed by using one 3-dimensional symmetric tensor h_{ij}, which satisfies:

$$h_i{}^i = 0, \quad h_{ij}{}^{,i} = 0. \tag{2.14}$$

Then, a general tensor metric perturbation is:

$$\delta^{(t)} g_{\mu\nu} = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}. \tag{2.15}$$

We notice that we have ten independent functions to describe metric perturbations, which is in fact the number of independent components of a 4-rank symmetric tensor.

The most important advantage of this classification for linear metric perturbations is that each one of them evolves independently. This allows us to study scalars, vectors and tensors separately, and so divide a long problem in several shorter ones.

In this section we have shown the main equations in the Big Bang model. We have also shown the standard classification for linear metric perturbations. The specific form for matter perturbations will be addressed later.

As we mentioned before, several questions are left unanswered by the Big Bang model. Some of these issues will be explained in the following section, along with a possible solution to them: inflation theory. Inflation has been very successful because it also makes a correct prediction for cosmic microwave background anisotropies and matter distribution today.

2.2 Inflation

Cosmic inflation is an early period of accelerated expansion of the Universe. It is believed to occur at an energy scale of 10^{15} GeV. This period is followed by the radiation-dominated era, dust era, and dark energy era. The very successful idea of inflation, originally developed in Guth (1981), offers a solution to the horizon problem and the flatness problem, among others (Garcia-Bellido (2000)). It also gives an explanation to the origin of primordial perturbations in the Universe, and makes a specific prediction for the power spectrum for primordial matter fluctuations: nearly scale-invariant. It also assumes Gaussian and adiabatic fluctuations. These predictions are in agreement with the observations (Komatsu et al. (2011)). There are many different inflation scenarios, but the most common scenario is the one where the accelerated expansion is driven by one scalar field with special dynamics.

Specifically, inflation is a quantum cosmological perturbation theory in which the Einstein-Hilbert action is coupled to a scalar field with a certain potential. In order to produce an accelerated expansion, a negative pressure for the scalar field is required. During inflation, the scale factor is expected to grow by a factor of 10^{28}, at least. The consequences are that all inhomogeneities are smoothed and the Universe flattens, which leads to a flat and large-scale homogeneous and isotropic Universe today. This accelerated expansion is controlled by the so-called slow-roll parameters, and the slow-roll approximation.

We will now explain the flatness and horizon problems, which are the original motivations to use the inflation theory. This will be followed by the explanation of the implementation with a single scalar field and the slow-roll approximation. Finally, we write a second-order action for the only physical scalar degree of freedom in this theory and apply the canonical quantisation in quantum field theory to find the power spectrum. The particle coming from the scalar field which drives inflation is called inflaton.

2.2.1 Flatness problem

As we mentioned previously, the Universe at large scales can be described by a homogeneous and isotropic metric with an expansion parameter depending on time. In general, a metric with these characteristics can be written as:

$$ds^2 = a(t)^2 \left[d\eta^2 - \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right], \qquad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{2.16}$$

This is called the Friedmann-Lemaître-Robertson-Walker metric. This metric is in polar coordinates and k is a constant parameter related to the curvature of the space. We can have $k < 0$, $k = 0$ and $k > 0$, for negative, zero and positive spatial curvature, respectively. On the other hand, the Friedmann equation for this metric coupled to a perfect fluid is:

$$\mathcal{H}^2 = \frac{1}{3}\rho a^2 - k, \tag{2.17}$$

where ρ is the energy density of the fluid in rest, and $\mathcal{H}$ the comoving Hubble parameter. A critical density ρ_c, needed to obtain a flat Universe, can be defined, i.e. if the total energy density of the perfect fluid is ρ_c, then the Universe has an Euclidean spatial geometry. The value for this critical density at a particular time t_* is:

$$\rho_c = \frac{3\mathcal{H}^2(t_*)}{a^2(t_*)}. \tag{2.18}$$

We can also define a density parameter Ω:

$$\Omega = \frac{\rho}{\rho_c}. \tag{2.19}$$

From here, we can see that $\Omega = 1$ represents flatness. If we combine (2.17), (2.18) and an equation of state $p = \omega\rho$, the density parameter satisfies:

$$\frac{d\Omega}{d\ln a} = \Omega(1 + 3\omega)(\Omega - 1), \tag{2.20}$$

from which we can see that Ω has an unstable equilibrium around $\Omega = 1$ for $\omega > -1/3$. Consequently, any deviation from flatness at early times is expected to be very large today.

Observations of the CMB anisotropies today say that the Universe is almost flat. In fact, the density parameter is estimated to be $0.9937 < \Omega < 1.0178$ (see Komatsu et al. (2011)). If the Universe is almost flat now, Ω must have been fine-tuned to 1 at early times. In fact, to reproduce the experimental value of Ω today, we would need $|\Omega - 1| \lesssim 10^{-55}$ at the GUT scale (Baumann (2007)). It seems, then, that the Universe had this particular initial condition. In the standard Big Bang model this precise initial condition must be assumed without explanation. The question is: how did the initial energy density come to take this value? This is called the flatness problem. An answer can be given by the inflation theory.

This is considered as a problem because it is more natural to think that the initial energy density of the Universe could take any value, in which case it would be more probable to have a non-flat Universe today (under the implicit assumption that all Ω values are equally likely at an early stage), and so it would be strange to have today the specific value ρ_c. However, some scientists think that this is not a problem at all (see Helbig (2012)).

2.2.2 Horizon problem

According to standard cosmology, photons decoupled from the rest of the components (baryons and electrons) at a temperature of the order of 3000 K. This corresponds to the so-called last-scattering surface, at a redshift near 1100. From the epoch of last scattering onwards, photons free-stream and reach us basically untouched. Detecting primordial photons is therefore equivalent to taking a picture of the Universe at the time of the last scattering. Observations of this cosmic background radiation show a spectrum consistent with that of a black body at temperature 2.725 K (except for small anisotropies of order

10^{-5}). This suggests that all these points were causally connected at the time of this surface, and as a consequence they reached a thermal equilibrium, which is observed today. The Big Bang model cannot explain this observation, and to know why we will need to study the evolution of the cosmological horizon.

The cosmological horizon is a surface in the space centred on a certain observer, such that the region inside this sphere is observable but the exterior is unobservable. This defines how large a region of the Universe can be in causal contact. The radius of this sphere is defined as the distance that a photon has traveled since the Big Bang. This means, two objects separated by a distance larger than the horizon today, have never been in causal contact. The cosmological horizon in a time t is

$$d_h(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_0^t \frac{da'}{a'^2} H'^{-1} \quad \Rightarrow \quad d_h \propto \frac{1}{H}, \tag{2.21}$$

where $1/H$ is the Hubble radius[1]. We interpret the Hubble radius as the distance at which objects have a recession velocity (according to the Hubble's law) equal to that of light. Since the cosmological horizon is proportional to the Hubble radius, it is enough to study the evolution of the Hubble radius. It is important to mention that all objects outside the Hubble radius are not in causal contact now, but it does not mean that they have never been in causal contact.

In the Big Bang model, one can study the evolution of the Hubble radius. In particular, we can compare the length corresponding to our present Hubble radius at the time of the last scattering to the size of the Hubble radius at that time. We would see that the latter is smaller that the former. So, at the time of last scattering, there were regions causally disconnected within the volume that is now our observable Universe. The question is: how can disconnected regions have the same temperature? This is the horizon problem.

[1]The fact that the cosmological horizon is proportional to the Hubble radius is true for the standard forms of energy considered in cosmology. These are such that $a \propto t^n$ or $a \propto e^{\alpha t}$, where n and α are any constant.

2.2.3 Solution

To explain the solution to these two problems, we define the comoving Hubble radius R_c:

$$R_c \equiv \frac{1}{aH} = \frac{1}{\dot{a}}. \tag{2.22}$$

If the comoving Hubble radius is to decrease, then $\dot{a}$ must increase:

$$\frac{d^2 a}{dt^2} > 0. \tag{2.23}$$

So, the expansion of the Universe must be accelerating. This is why this period is called inflation.

The solution to the horizon problem is that, early on, the Universe might not have been dominated by radiation. Perhaps, it was dominated by another form of energy, which caused the comoving Hubble radius to decrease. In this way, at the time of the last scattering, the comoving Hubble radius could have been larger than ours today, so the bath of photons within the region of our observable Universe today could have reached a thermal equilibrium at that time. This period of inflation, would have to be followed by the radiation-dominated era, the matter-dominated era, and the dark energy era.

We expect to have roughly the behaviour shown in Figure 2.1 for the comoving Hubble radius during the early Universe. Figure 2.1 shows the evolution of the comoving Hubble radius R_c (green line) as a function of the scale factor during inflation and the radiation era. During inflation, the comoving Hubble radius decreases, and later it increases during radiation era. Therefore, a given comoving scale λ (blue line), say, the comoving distance between two galaxies, is first a sub-Hubble scale ($\lambda < R_c$), and there is causal contact between these two galaxies. Later, it is a super-Hubble scale ($\lambda > R_c$), and the galaxies cannot communicate. Finally, in the radiation era or later, it is again a sub-Hubble scale, and they have contact again. If λ corresponded to an observable scale for us today, it should have been smaller than the comoving Hubble radius scale at the last scattering time or previously in order to solve the horizon problem. Typically, inflation is considered to end at energy scales of order 10^{15} GeV. If so, the comoving Hubble radius at the end of inflation is estimated to be at least 28 orders of magnitude smaller than it is today. As a consequence, during inflation the scale factor should have grown at least 28 orders of magnitude. This is the minimum amount of inflation required to solve the horizon

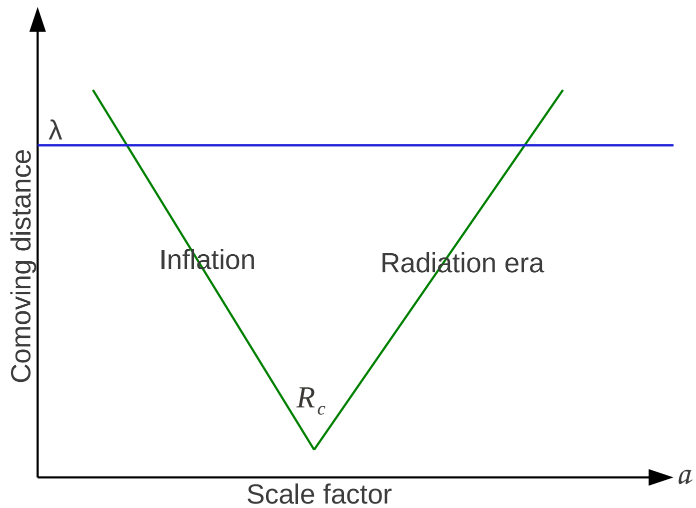

Figure 2.1: The comoving Hubble radius R_c (green line) as a function of the scale factor for early times: epoch of inflation and radiation. Very early on, a scale of interest λ (blue line) was smaller than the comoving Hubble radius and therefore physical processes were happening at that scale.

problem, but inflation could in fact go on for much longer.

The described behaviour of R_c in the inflation era could happen if the Universe is dominated by a form of energy such that:

$$p < -\frac{\rho}{3}. \tag{2.24}$$

Since the energy density is always positive, the pressure must be negative. This can be seen from the Einstein equation (2.7):

$$\frac{d^2a/d\eta^2}{a^3} = \frac{1}{6}(\rho - 3p) \quad \Rightarrow \quad \frac{d^2a/dt^2}{a} = -\frac{1}{6}(\rho + 3p). \tag{2.25}$$

Then, the condition (2.23) is equivalent to

$$\rho + 3p < 0 \quad \Rightarrow \quad p < -\frac{\rho}{3}. \tag{2.26}$$

In addition, this assumption of a period with accelerated expansion solves the flatness problem. Due to the condition (2.24), during inflation any deviation from $\Omega = 1$ tends to get smaller. Therefore, an arbitrary value of Ω before inflation could evolve to be as near to 1 as we want, depending on how long the inflation period was, which makes it more natural to expect $|\Omega - 1| \lesssim 10^{-55}$ after inflation. Inflation does not predict an exact value for Ω at early times (so it does not determine the parameter k), but it lets us have initial conditions with a large uncertainty before inflation and still have $\Omega \approx 1$ today.

2.2.4 Single scalar field and slow roll approximation

The most common implementation of inflation, with the requirement (2.24), is done with one scalar field $\varphi(\vec{x}, t)$ minimally coupled with gravity. The stress-energy tensor is

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu} \frac{\partial \varphi}{\partial x^{\nu}} \frac{\partial \varphi}{\partial x^{\beta}} - g^{\alpha}{}_{\beta} \left[\frac{1}{2} g^{\mu\nu} \frac{\partial \varphi}{\partial x^{\mu}} \frac{\partial \varphi}{\partial x^{\nu}} + V(\varphi) \right], \tag{2.27}$$

where $V(\varphi)$ is a potential for the field φ. Let us consider the field $\varphi(\vec{x}, t)$ coupled to the metric in (2.1). In that case, φ will only depend on time, and the energy density and pressure will be

$$\rho = T^{0}{}_{0} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{2.28}$$

$$p = T^{i}{}_{i} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{2.29}$$

where $\dot{\varphi}$ represents the derivative of φ with respect to the physical time t. It follows that negative pressure occurs when there is more potential energy than kinetic. This happens if $\dot{\varphi} \approx 0$ while $V(\varphi)$ is nonzero.

Figure 2.2 shows a model for the potential $V(\varphi)$. It has a very flat zone in which $\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi)$, where the scalar field slowly rolls toward its ground state. This assumption in the form of $V(\varphi)$ is called slow-roll approximation. In order to quantify the flat zone, two parameters are defined:

$$\epsilon_V \equiv \frac{1}{2} \left(\frac{V'}{V} \right)^2, \tag{2.30}$$

$$\delta_V \equiv \frac{V''}{V}. \tag{2.31}$$

These are called slow-roll parameters. Here $V' \equiv \frac{dV}{d\varphi}$ and $V'' \equiv \frac{d^2V}{d\varphi^2}$. The subindex V is used here because these slow-roll parameters are defined only in terms of the potential

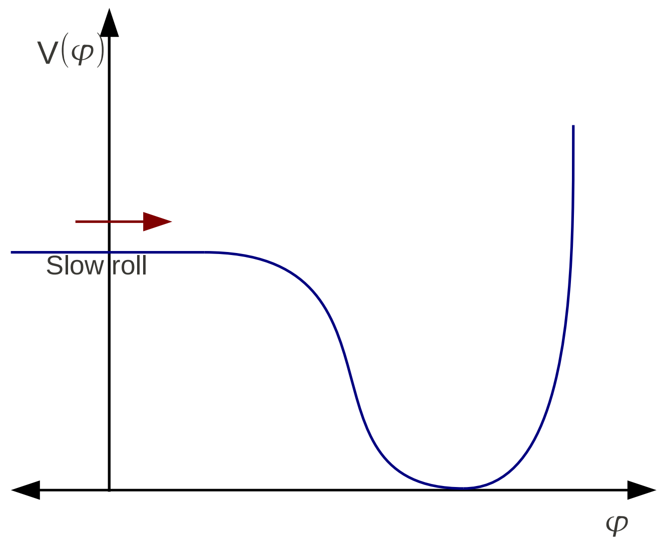

Figure 2.2: Example of $V(\varphi)$. It contains a very flat section in which inflation begins. The scalar field slowly rolls to the right, toward its ground state.

$V(\varphi)$, but in literature different definitions can be found (see, for example, Dodelson (2003)).

When these parameters are small, the shape of the potential is restricted and the slow-roll approximation holds. In that case, in the equation (2.6), we can neglect the kinetic term $\dot{\varphi}$, and we can approximate

$$3H^2 \approx V(\varphi). \tag{2.32}$$

In (2.8), we neglect the term $\ddot{\varphi}$, obtaining

$$3H\dot{\varphi} + V' \approx 0. \tag{2.33}$$

Using these last two equations, we can approximate

$$\epsilon_V \approx \frac{1}{2}\frac{\dot{\varphi}^2}{H^2} = -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right), \tag{2.34}$$

$$\delta_V \approx -\frac{1}{H}\frac{\ddot{\varphi}}{\dot{\varphi}} + \frac{1}{2}\frac{\dot{\varphi}^2}{H^2}. \tag{2.35}$$

These approximations hold when the slow-roll parameters are small. They will be useful to simplify some calculations in the quantum theory by considering only relevant terms during inflation. From (2.34) we can see that under the slow roll approximation, H is almost constant. Therefore, the metric describes almost a de Sitter space.

2.2.5 Matter perturbations

Since inflation is apparently a good description for the beginning of the Universe, we will use it to study primordial cosmological perturbations.

In the previous section we did not specify matter perturbations. Consider perturbations for the scalar field described above:

$$\varphi = \varphi^{(0)}(t) + \varphi^{(1)}(\vec{x}, t), \quad |\varphi^{(1)}| \ll |\varphi^{(0)}|. \tag{2.36}$$

Then, the stress-tensor perturbations in conformal coordinates:

$$T^{(1)0}{}_0 = \frac{1}{a^2}\left[-\varphi^{(0)\prime 2}\varphi^{(1)} + \varphi^{(0)\prime}\varphi^{(1)\prime} + V'a^2\varphi^{(1)}\right], \tag{2.37}$$

$$T^{(1)0}{}_i = \frac{1}{a^2}\varphi^{(0)\prime}\varphi^{(1)}_{,i}, \tag{2.38}$$

$$T^{(1)i}{}_j = \left[\varphi^{(0)\prime 2}\varphi^{(1)} - \varphi^{(0)\prime}\varphi^{(1)\prime} - V'a^2\varphi^{(1)}\right]\delta^i_j, \tag{2.39}$$

where V' is evaluated at $\varphi^{(0)}$.

With the classification of scalar, vector and tensor perturbations explained above, we can see that matter perturbations have only scalar perturbations. This means, $\varphi^{(1)}(\vec{x}, t)$ will be coupled only to scalar metric perturbations.

In this section we have shown the main picture of the theory of inflation. We described the two main problems solved by inflation and the most common implementation: with a single scalar field. We also considered first order perturbations in inflation, showing the explicit form of the stress-energy tensor perturbations.

Having described all the primordial first order perturbations in General Relativity with inflation, the next step is to study the quantum theory of these primordial fluctuations. In order to do this, we must quantise the fluctuations and calculate their 2-point Green functions. This process has some subtleties, one of which is related to the gauge invariance of this theory. For this reason, before talking about quantum fluctuations, we shall continue with a complete section of discussion about the gauge invariance.

2.3 Gauge invariance

The action for General Relativity with inflation is the following:

$$S = -\frac{1}{2}\int R\sqrt{-g} + \int \sqrt{-g}\left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right), \tag{2.40}$$

which has a gauge symmetry that comes from considering general coordinate transformations. However, in the linear perturbation theory, this action is invariant only under infinitesimal transformations of first order. Then, all concepts are restricted to first order perturbations. In particular, the gauge-invariant variables used here are invariant only to first order.

Let us study the infinitesimal coordinate transformations. Suppose we perform the following change of coordinates:

$$\tilde{x}^\mu = x^\mu + \xi^\mu, \tag{2.41}$$

where $\xi^\mu = \xi^\mu(x)$ is an arbitrary infinitesimal shift vector function. We split ξ^μ into two parts:

$$\xi^\mu = (\xi^0, \xi^i), \quad \text{such that} \quad \xi^i \equiv \xi^{iT} + \partial_i\xi \quad \text{and} \quad \partial_i\xi^{iT} = 0. \tag{2.42}$$

Here, we shift the three-spatial indices with the three-spatial metric δ_{ij} and its inverse δ^{ij}. As we mentioned before, tensor, vector and scalar parts are decoupled. Thus, the vector gauge freedom ξ^{iT} (vector part of the shift vector) is related only to vector perturbations of the metric. Similarly, the scalar gauge freedoms ξ^0 and ξ (scalar parts of the shift vector) are related only to scalar perturbations of the metric. Since there is no tensor part of the shift vector, tensor metric perturbations h_{ij} are gauge-invariant.

From now on, we will focus on scalar perturbations, so let us see how the scalar metric and matter perturbations transform under this infinitesimal coordinate transformation. In general, metric transforms from g to $\tilde{g}$ such that:

$$\Delta g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\xi_{\nu;\mu} - \xi_{\mu;\nu}. \tag{2.43}$$

Considering the background (2.1) and scalar perturbations (2.11), equation (2.43) gives

$$\tilde{\phi} = \phi - (a'/a)\xi^0 - \xi^{0\prime}, \quad \tilde{\psi} = \psi + (a'/a)\xi^0, \quad \tilde{B} = B + \xi^0 - \xi', \quad \tilde{E} = E - \xi. \tag{2.44}$$

The scalar perturbation for inflation also transforms:

$$\tilde{\varphi}^{(1)} = \varphi^{(1)} - \varphi^{(0)\prime}\xi^0. \tag{2.45}$$

The arbitrariness of the vector ξ^μ is called gauge freedom, and it can cause some confusions. Not all apparently perturbed metrics are real perturbed space-times. For instance, a homogeneous and isotropic form of the metric can be transformed to an inhomogeneous form by performing a particular coordinate transformation. So, in order to be certain whether we are considering a homogeneous and isotropic space-time or a perturbed one, we will need to distinguish physical inhomogeneities (geometrical) and coordinate artifacts.

One approach to this problem is to work in a manifestly gauge-invariant framework. It consists in defining a new set of gauge-invariant perturbation fields and rewrite all the actions (or equations of motion) in terms of these new fields. Since, in general, not all the gauge-invariant perturbation fields are physical, one must also find the physical degrees of freedom by finding only relevant gauge-invariant fields which the action depends on. This method is easier to work with than others because all physical quantities are gauge-invariant. Specifically, we are going to follow the idea of (Maldacena (2003)), which consists in fixing all the gauge freedoms and eliminating non-physical perturbation fields, in such a way that the final physical degrees of freedom are gauge-invariant.

2.3.1 Gauge choice

We have seen that there are two gauge scalar degrees: ξ^0 and ξ. Since they are arbitrary, we shall use them to set some scalar perturbations to zero. It is necessary to be very careful with our gauge choice because we will study quantum cosmological perturbations. In the classical theory of linear perturbations, one works only with the equations of motion, and one could fix the gauge freedom by setting any pair of fields to be zero, consistent with (2.44) and (2.45), without losing neither information nor generality. However, in the quantum perturbation theory, one works with the action, with which it is only possible to set fields with redundant equations to zero. This is because when a field is eliminated in the lagrangian, its equation of motion will not appear, and its information will consequently be lost. If the gauge freedom is fixed in an action by setting to zero fields which do not have a redundant equation of motion, crucial information contained in these equations of motions would be lost. The resulting action with the gauge fixed would be different to that without the gauge fixed. Since redundant equations of motion do not carry any new information, there is no problem in eliminating them.

How can we easily find a pair of fields with redundant equations of motion? This can be done by studying the infinitesimal gauge invariance of the perturbed action (2.40). This perturbed action is obtained by replacing the perturbations fields in (2.40) and expanding the Taylor series up to second-order in these fields. This action gives first order equations for each field. Considering only scalar perturbation fields, a general variation of the perturbed action is:

$$\delta S = \int \left(Eq_\phi \delta\phi + Eq_\psi \delta\psi + Eq_E \delta E + Eq_B \delta B + Eq_{\varphi^{(1)}} \delta\varphi^{(1)} \right), \tag{2.46}$$

where Eq_n denotes the equation of motion for a field n. But if we replace these variations of fields δn by the gauge variations (2.44)-(2.45), we should obtain $\delta S = 0$, because the action is gauge-invariant. Let us do this and perform some integration by parts:

$$\delta S_{\text{gauge}} = \int \left(Eq'_\phi + (Eq_\psi - Eq_\phi)\frac{a'}{a} + Eq_B \; - Eq_{\varphi^{(1)}}\varphi^{(0)'} \right)\xi^0 + \left(Eq'_B - Eq_E \right)\xi. \tag{2.47}$$

Since $\delta S_{\text{gauge}} = 0$, both parenthesis are zero because ξ and ξ^0 are completely arbitrary. This gives us two relations between the equations of motion:

$$Eq'_\phi + (Eq_\psi - Eq_\phi)\frac{a'}{a} + Eq_B - Eq_{\varphi^{(1)}}\varphi^{(0)'} = 0, \tag{2.48}$$

$$Eq'_B - Eq_E = 0. \tag{2.49}$$

Any field that has an equation which can be worked out from these last two equations, has, of course, a redundant equation of motion. Thus, the possibilities to fix the gauge in the action are:

$$(\psi, \varphi^{(1)}) + (E). \tag{2.50}$$

Equation (2.50)[2] means that we can use one scalar gauge freedom to set the value of

[2]This equation does not include B as a possible field to be fixed by the gauge choice. Its equation of motion can, in fact, be worked out from (2.48) but the problem is that B cannot be fixed along with

one field from the first parenthesis and the other gauge freedom to set one field from the second parenthesis.

Our gauge choice will be:

$$\tilde{\varphi^{(1)}} = \tilde{E} = 0. \tag{2.51}$$

These conditions fix ξ and ξ^0 uniquely:

$$\xi = E \quad \text{and} \quad \xi^0 = \varphi^{(1)}/\varphi^{(0)}. \tag{2.52}$$

Now we have reduced our initial problem with 5 scalar perturbation fields to one with 3: ψ, B and ϕ. This will simplify our later work. This gauge choice is good because if we define the following gauge-invariant field:

$$\mathcal{R} \equiv \psi + \frac{\mathcal{H}}{\varphi^{(0)\prime}}\varphi^{(1)}, \tag{2.53}$$

called comoving curvature perturbation, it turns out to be identical to the perturbative metric variable ψ:

$$\mathcal{R} = \psi \quad \text{if} \quad \varphi^{(1)} = 0. \tag{2.54}$$

Thus, in this gauge choice ψ is gauge-invariant, but the other fields left are not necessarily so. We will see later that $\mathcal{R}$ is the only physical scalar field in the theory of inflation.

In this section we have discussed the difficulties related to the gauge freedom in the theory of inflation. We described the framework we will use later: gauge-invariant method. We showed an explicit way to fix the gauge freedom in an action, along with the choice we will use in the following section.

The next step is to study the quantum perturbation theory for inflation. That means, to write explicitly a second-order action for the only physical degree of freedom in this theory, quantise it, and calculate its two-point Green function.

E. This is because the information in the equation for E is contained in the equation for B (see 2.49). Thus, if E is eliminated from the action we must keep B to not lose information.

2.4 Quantum inflation theory

As we have said before, although the cosmic microwave background indicates that the Universe in the past was extraordinarily homogeneous and isotropic, we know that the Universe today is not exactly homogeneous: we observe galaxies, clusters and superclusters on large scales. These structures are expected to arise from very small quantum primordial inhomogeneities that grow in time via gravitational instability, and that may have originated tiny ripples in the metric and matter. Those ripples must have left some trace as temperature anisotropies in the microwave background. In fact, such anisotropies were discovered by the COBE satellite in 1992. They appear as perturbations in temperature of only one part in 10^5. Also, these ripples originated observed structures today, through gravitational collapse.

Furthermore, the anisotropies observed by WMAP are in agreement with a small-amplitude, nearly scale-invariant, primordial power spectrum of inhomogeneities (Komatsu et al. (2011)). The power spectrum is defined as the Fourier transformation of the spatial 2-point correlation function. In addition, in order to explain the distribution of galaxies and clusters of galaxies on very large scales in our observable Universe, a scale-invariant density perturbation spectrum was proposed (Harrison (1970)).

Besides the two problems solved by inflation and described previously, one of the most astonishing predictions of inflation is that quantum fluctuations of the inflationary field generate large-scale perturbations in the metric and matter, predicting a nearly scale-invariant power spectrum for primordial perturbations, as observed. For this reason, quantum linear perturbation theory with inflation has been very successful.

The quantum inflation theory consists in quantising the action (2.40) for the physical perturbation fields and studying its properties. Focusing on scalar type perturbations, the relevant action is a second-order action in the perturbation fields obtained by taking (2.40), replacing (2.36) and (2.11), and expanding in Taylor series up to second-order. As we explained in the previous section, we will fix the gauge (2.51) to simplify this second-order action. Also, we will eliminate all non-physical degrees of freedom.

We expect to have only 1 scalar physical degree of freedom. It is known that General Relativity without matter has no scalar degree of freedom, and the matter action for inflation has 1 scalar degree of freedom $\varphi^{(1)}$. In total, then, there is only 1 scalar degree of freedom. As we said in the previous section, we will describe it with ψ.

Tensor perturbations are also relevant, because they can be, in principle, observed today. The inflation theory also predicts a nearly scale-invariant power spectrum for tensor perturbations. A detailed analysis of quantum tensor perturbations can be found in Dodelson (2003).

In this section we explicitly calculate an action of second-order in the scalar perturbations for matter and metric, and an action for the only physical scalar field. We quantise this field and calculate its power spectrum. We also mention the result of the power spectrum for tensor perturbations.

2.4.1 Second-order action

Let us start considering the complete action:

$$S = -\frac{1}{2}\int R\sqrt{-g} + \int \sqrt{-g}\left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right). \tag{2.55}$$

First, we replace here the perturbed metric and matter fields, with our gauge choice (2.51):

$$ds^2 = a^2[(1+2\phi)d\tau^2 - 2B_{,i}dx^i d\tau - [(1-2\psi)\delta_{ij}]dx^i dx^j], \tag{2.56}$$

$$\varphi = \varphi^{(0)}. \tag{2.57}$$

Second, we expand it up to second-order in the perturbation variables, performing a Taylor series, to get:

$$\begin{aligned} S^{(2)} = \frac{1}{2}\int d^3x d\eta a^2 \Big[& -6\psi'^2 - 12\mathcal{H}(\phi+\psi)\psi' - 9\mathcal{H}^2(\phi+\psi)^2 - 2\psi_{,i}(2\phi_{,i} - \psi_{,i}) \\ & - 4\mathcal{H}(\phi+\psi)B_{,ii} - 4\psi' B_{,ii} - 4\mathcal{H}\psi_{,i}B_{,i} + 3\mathcal{H}^2 B_{,i}B_{,i} \\ & + \left(-\frac{1}{2}\phi^2 - 3\phi\psi + \frac{3}{2}\psi^2 + \frac{1}{2}B_{,i}B_{,i}\right)\left(\varphi^{(0)\prime 2} - 2Va^2\right) \\ & + \left(2\phi^2 - B_{,i}B_{,i} + 6\psi\phi\right)\varphi^{(0)\prime 2}\Big]. \end{aligned} \tag{2.58}$$

This action gives first order equations of motion for the perturbative scalar fields. In the calculation of this action, all zero and first-order terms vanish due to the background equations of motion.

Next, we will eliminate the non-physical degrees of freedom in this action.

2.4.2 Reduced second-order action

From now on, we are going to work in the Fourier space:

$$\phi(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\phi(\eta,\vec{k})e^{i\vec{x}\cdot\vec{k}}, \tag{2.59}$$

$$\psi(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\psi(\eta,\vec{k})e^{i\vec{x}\cdot\vec{k}}, \tag{2.60}$$

$$B(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}B(\eta,\vec{k})e^{i\vec{x}\cdot\vec{k}}, \tag{2.61}$$

and we define $k^2 \equiv \vec{k}\cdot\vec{k}$. For simplicity, we will omit the arguments of the fields, but we must keep in mind that we are in Fourier space, so fields depend on $(\eta,\vec{k})$.

From equation (2.58), we can see that neither B nor ϕ have time derivatives. Thus, they are auxiliary variables which can be worked out from their equations of motion in terms of ψ, obtaining:

$$\phi = -\frac{\psi'}{\mathcal{H}}, \tag{2.62}$$

$$B = \frac{2\mathcal{H}k^2\psi + \psi'\varphi^{(0)\prime 2}}{2\mathcal{H}^2k^2}. \tag{2.63}$$

Replacing the expressions (2.62)-(2.63) into the action, we obtain the final second-order action for ψ:

$$S^{(2)} = \frac{1}{2}\int d^3k d\eta \frac{a^2\varphi^{(0)\prime 2}}{\mathcal{H}^2}\left[\psi'^2 - k^2\psi^2\right], \tag{2.64}$$

where we have used some background equations to simplify. This action depends on ψ, which is a gauge-invariant variable and corresponds to the only physical scalar field in inflation. Note that, as a consequence of the spatial isotropy of the background space-time, the function ψ does not depend on the direction of $\vec{k}$, but only on its magnitude.

Now that we have an action for the physical field, we would like to find the quantum solution for ψ. In order to do this, we will apply the standard QFT rules of canonical quantisation. First we will define a new gauge-invariant field $u(\eta,\vec{k})$:

$$u = z\psi, \quad \text{where} \quad z \equiv a\frac{\varphi^{(0)\prime}}{\mathcal{H}}, \tag{2.65}$$

such that the second-order action in, terms of u, has the form for a canonical scalar field in Minkowski space with a time-dependent mass:

$$S^{(2)} = \frac{1}{2}\int d^3k d\eta \left[u'^2 - (k^2 - \frac{z''}{z})u^2\right]. \tag{2.66}$$

The time dependence of the mass is due to the interactions with the expanding background. The corresponding equation of motion for u is:

$$u'' + (k^2 - \frac{z''}{z})u = 0. \tag{2.67}$$

Let us start now with the quantisation process.

2.4.3 Quantisation

In order to find an analytic solution for the quantum field u, we will use the slow roll approximation.

The first step in the canonical quantisation is to obtain the momentum π canonically conjugated to the field u:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial u'} = u'. \tag{2.68}$$

Second, we promote π and u to operators $\hat{\pi}$ and $\hat{u}$, respectively. They must satisfy the following commutation relations at a given time η:

$$[\hat{u}(\eta,\vec{x}),\hat{u}(\eta,\vec{x}')] = [\hat{\pi}(\eta,\vec{x}),\hat{\pi}(\eta,\vec{x}')] = 0, \quad [\hat{\pi}(\eta,\vec{x}),\hat{u}(\eta,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}'), \tag{2.69}$$

where we have used $\hbar = 1$. Also, $\hat{u}(\eta,\vec{k})$ must satisfy the classical equation of motion:

$$\hat{u}'' + (k^2 - \frac{z''}{z})\hat{u} = 0. \tag{2.70}$$

Operators $\hat{u}(\eta,\vec{k})$ and $\hat{u}(\eta,\vec{x})$ are related by the Fourier transformation:

$$\hat{u}(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\hat{u}(\eta,\vec{k})e^{\vec{x}\cdot\vec{k}}. \tag{2.71}$$

In order to find the solutions $\hat{u}$ and $\hat{\pi}$, satisfying (2.69) and (2.70), we expand them as it is usually done in quantum field theory:

$$\hat{u}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(u_{\vec{k}}(\eta) a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + u_{\vec{k}}(\eta)^* a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right), \quad (2.72)$$

$$\hat{\pi}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(u_{\vec{k}}(\eta)' a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + u_{\vec{k}}(\eta)^{*\prime} a^\dagger_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right), \quad (2.73)$$

where $\{u_{\vec{k}}(\eta), u^*_{\vec{k}}(\eta)\}$ is a complete base for solutions of equation (2.67), and $a_{\vec{k}}$ with $a^\dagger_{\vec{k}}$ are the annihilation and creation operators for a bosonic field, respectively. Then,

$$[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = [a_{\vec{k}}, a_{\vec{k}'}] = 0, \quad [a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k} - \vec{k}'). \quad (2.74)$$

If we replace (2.72)-(2.73) into (2.69), we find they are consistent with the commutation relations (2.74) only if the mode functions $u_{\vec{k}}$ obey the normalization conditions:

$$u'_{\vec{k}} u^*_{\vec{k}} - u_{\vec{k}} u'^*_{\vec{k}} = i. \quad (2.75)$$

In addition, we define the Fock space as the set of states that corresponds to successive applications of creation operators to a vacuum state. The vacuum state $|0\rangle$ is defined by the following condition:

$$a_{\vec{k}}|0\rangle = 0. \quad (2.76)$$

The vacuum state is not defined until the solution $u_{\vec{k}}$ is completely specified. From (2.75), it follows that $u_{\vec{k}}$ is a complex solution of a second-order differential equation. This means, in principle, it contains 4 undetermined constants. One of these constants is irrelevant because it corresponds to a non-physical phase factor. Another constant is determined by the condition (2.75). Then, for each $\vec{k}$ there remain two arbitrary real numbers that are unspecified. Thus, to specify $u_{\vec{k}}$ completely, we just have to find an initial condition or, equivalently, a specific normalization. For a free scalar field with constant mass in Minkowski, we fix $u_{\vec{k}}$, by adding the condition of minimising the vacuum energy (hamiltonian). Since our mass is time-dependent, the hamiltonian is not conserved in time, so we must give an initial condition at a given time.

In general, we define different vacua if we quantise in different coordinates and/or if we minimise the hamiltonian at different times. Examples of the arbitrariness of the vacuum in different coordinates are the Unruh effect and the Hawking radiation. In these examples we can see that the vacuum defined in one coordinate system, corresponds to a state containing a non-zero number of particles in the other coordinate system. The two different vacua are related by the Bogoliubov transformations (see Mukanov (2004) for more information).

From this point of view, we could define any state we wish to be the vacuum and build a completely consistent quantum field theory. Then we must ask: which is the physical vacuum state? To define the real vacuum, we will follow the standard QFT. There, the vacuum is the zero-particle state (state with minimum energy) seen by an inertial observer (in Minkowski). In the expanding Universe case, the vacuum will be the one defined in conformal coordinates, because this space-time is just a Minkowski space with a time-dependent conformal factor in front. This vacuum state is the zero-particle state seen by a geodesic observer, that is, one in free-fall in the expanding space. As we said before, the hamiltonian in our case is not constant. So, we will define the zero-particle state, as the one which minimises the hamiltonian only in the regime of short wavelengths ($k/a \gg H$). This is referred to as the Bunch-Davies vacuum, and it is justified because only early short-wavelength states are observable today. Early long-wavelength states are stretched in time to huge, unobservable scales. Also, as we will see later, in the short-wavelength regime the solution is oscillating and then the choice of vacuum is clear.

Finally, the problem is reduced to find the functions $u_{\vec{k}}$ and their corresponding normalisation. However, instead of determining the scalar field solution in general inflation, we will use the well-known solution in de Sitter inflation. We will promote relevant parameters on which this spectrum depends to slowly varying functions of time, according to the slow roll approximation.

From eq. (2.34), we can see that if $\epsilon_V = 0$, then H is constant. This justifies the fact that we are going to find the solution of $u_{\vec{k}}(\eta)$ in de Sitter space and use it as a first approximation of inflation. Also, we can find that

$$\frac{z^{''}}{z} = \frac{a^{''}}{a}\left(1 + \mathcal{O}(\epsilon_v) + \mathcal{O}(\delta_V)\right). \tag{2.77}$$

So, in de Sitter space, the equation of motion for $u_{\vec{k}}$ is:

$$u^{''} + \left(k^2 - \frac{a^{''}}{a}\right) u = 0. \tag{2.78}$$

Also, since H is constant,

$$H = H_0 \quad \Rightarrow \quad a = \frac{-1}{H_0 \eta} \quad \Rightarrow \quad \frac{a''}{a} = \frac{2}{\eta^2} = 2H_0^2 a^2, \tag{2.79}$$

where H_0 is the constant value of the Hubble parameter in de Sitter space. Then, (2.78) becomes

$$u'' + \left(k^2 - 2H_0^2 a^2\right) u = 0. \tag{2.80}$$

In the short-wavelength regime $k \gg H_0 a$ (equivalently $k|\eta| \gg 1$), this equation turns out to be

$$u'' + k^2 u \approx 0. \tag{2.81}$$

In this equation, the choice of vacuum is clear. Its properly normalised solution is

$$u_{\vec{k}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}. \tag{2.82}$$

Similarly, the properly normalised solution $u_{\vec{k}}$ to eq. (2.80) is:

$$u_{\vec{k}}(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{2.83}$$

This solution satisfies (2.75) and has the normalisation (2.82) in the short-wavelength regime. This solution defines unambiguously the Bunch-Davies vacuum state $|0\rangle$. Notice that in the asymptotic past $\eta \to -\infty$, we recover the vacuum solution (2.82).

Now that we have found the quantum solution for the only scalar degree of freedom in de Sitter space, the next step is to calculate its spectrum, and promote relevant parameters in order to find the spectrum during inflation.

2.4.4 Spectrum of perturbations

We continue focusing on scalar perturbations. We are interested in finding the spectrum $\mathcal{P}_u(\eta, k)$ of the quantum field $u_{\vec{k}}$. The two-point Green function of $\hat{u}(\eta, \vec{x})$ can be calculated in terms of $u_{\vec{k}}$ using (2.71) and (2.72):

$$\begin{aligned} \langle 0|\hat{u}^\dagger(\eta,\vec{x})\hat{u}(\eta,\vec{x}')|0\rangle &= \int \frac{d^3k}{(2\pi)^3}|u_{\vec{k}}|^2 e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}, \\ &\equiv \int \frac{dk}{k}\mathcal{P}_u(\eta,k)\frac{\sin(kr)}{kr}, \end{aligned} \tag{2.84}$$

where $r \equiv ||\vec{x}-\vec{x}'||$, and the scalar field power spectrum is:

$$\mathcal{P}_u(\eta,k) = \frac{k^3}{2\pi^2}|u_{\vec{k}}(\eta)|^2. \tag{2.85}$$

Replacing the solution (2.83), we obtain

$$\mathcal{P}_u(\eta,k) = \frac{k^2}{4\pi^2}\left(1+\frac{1}{(k\eta)^2}\right). \tag{2.86}$$

The function $\mathcal{R}$:

$$\mathcal{R} = \psi + \mathcal{H}\frac{\varphi^{(1)}}{\varphi'^{(0)}} \tag{2.87}$$

is gauge-invariant and is related to the gauge-dependent spatial curvature perturbation ψ on a generic slicing to the inflaton perturbation $\varphi^{(1)}$ in that gauge. By construction, $\mathcal{R}$ represents the gravitational potential on comoving hypersurfaces where $\varphi^{(1)} = 0$, i.e. $\mathcal{R} = \psi$. This scalar field is useful because, as we will see, it freezes for super-horizon scales, letting us connect the spectrum of inflation with the one during radiation/matter dominated era.

Now we go from de Sitter to inflation by promoting H_0 to a slowly varying function of time. Following our previous calculations, the relation between u and $\mathcal{R}$ is given by (2.65):

$$u_{\vec{k}} = a\sqrt{2\epsilon_V}\mathcal{R}_{\vec{k}}, \tag{2.88}$$

here we have used the relation (2.34). Then, the spectrum for the scalar field $\mathcal{R}$ during inflation is:

$$\mathcal{P}_{\mathcal{R}} = \frac{H^2}{8\pi^2\epsilon_V}\left(1+\frac{k^2}{(Ha)^2}\right). \tag{2.89}$$

Notice that for a given comoving momentum k, the spectrum in the super-Hubble scales is nearly constant:

$$\mathcal{P}_{\mathcal{R}} = \frac{H^2}{8\pi^2\epsilon_V}; \quad k/a \ll H. \tag{2.90}$$

We say that the spectrum freezes after the first horizon crossing. One can demonstrate more rigorously that $\mathcal{R}' \approx 0$ in super-Hubble scales (Dodelson (2003)), in inflation and radiation/matter eras. This characteristic allows us to connect the perturbations during inflation with the perturbations during radiation/matter era. Since the spectrum is almost constant, we can approximate it to its value at the time t_* of the horizon crossing:

$$\mathcal{P}_{\mathcal{R}} = \frac{H_*^2}{8\pi^2 \epsilon_V}; \quad k/a_* = H_*. \tag{2.91}$$

It is important to say that different modes cross back throught the horizon at different times, so they have different amplitudes. The dependence on t_* in the spectrum, brings a dependence on the momentum k. To show explicitly this dependence on k, we make a power-law approximation, and we write the spectrum as:

$$\mathcal{P}_{\mathcal{R}} = \frac{H_*^2}{8\pi^2 \epsilon_V} \left(\frac{k}{a_* H_*} \right)^{n_{\rm s}-1}, \tag{2.92}$$

where $n_{\rm s}$ is called spectral index, and is calculated from:

$$n_{\rm s} - 1 = \left. \frac{d\ln(\mathcal{P}_{\mathcal{R}})}{d\ln(k)} \right|_{k=a_* H_*} = 2(\eta_V - 3\epsilon_V), \tag{2.93}$$

where we have made use of the definition of the two slow roll parameters. The final quantum spectrum for $\mathcal{R}$ predicted by inflation is written as:

$$\mathcal{P}_{\mathcal{R}} = A_{\mathcal{R}}^2 k^{n_{\rm s}-1}. \tag{2.94}$$

Then, inflation predicts a nearly scale-invariant spectrum, with a small deviation given by $n_{\rm s} - 1$. The amplitude of the power spectrum $A_{\mathcal{R}}$ and the scalar spectral index $n_{\rm s}$ are not determined by the theory because they depend on the slow roll approximation. They are considered as cosmological parameters to be fixed by observations. In fact, observational values for $A_{\mathcal{R}}$ and $n_{\rm s}$ can be found in Komatsu et al. (2011).

In Figure (2.1) we can see that, after the first horizon crossing, the perturbation re-enters the horizon during radiation or matter dominated eras. This event marks the transition from quantum theory to classical theory. By making use of $\mathcal{R}$, it is possible to calculate the power spectrum of the invariant gravitational potential Φ (see Mukanov, Feldman and Brandenberger (1992)) when this second horizon-crossing occurs (see Riotto (2002)):

$$\mathcal{P}_\Phi = \begin{cases} \left(\frac{2}{3}\right)^2 \mathcal{P}_\mathcal{R} & \text{if perturbation re-enters in Radiation-Dominated era} \\ \left(\frac{3}{5}\right)^2 \mathcal{P}_\mathcal{R} & \text{if perturbation re-enters in Matter-Dominated era} \end{cases} \tag{2.95}$$

which is also scale-invariant. This power spectrum for Φ after inflation is what really matters because it is responsible for the energy density fluctuations, through the Poisson equation. Value (2.95) is used as an initial condition for the perturbations in the study of their late classical evolution (Dodelson (2003)).

On the other hand, in the linear theory of perturbation, the two-point function suffices to define all the higher-order even correlation functions, while the odd correlation functions vanish. Since the only prediction we have is the two-point function, inflation predicts only Gaussian processes. This means that after the transition to a classical field, the inflationary field is considered as a Gaussian random field: Fourier modes evolve independently. From the central limit theorem of statistics it follows that the probability density distribution of this classical field at any point in space is Gaussian, and it depends only on the power spectrum. Then, only one function–the power spectrum at horizon re-entry is required to specify all of the statistical properties of the initial density distribution (Dodelson (2003)). Non-Gaussian effects can be studied by considering second-order perturbation theory (see Maldacena (2003)).

In order to make testable predictions with inflation it is necessary to study the distribution of particles when the inflaton decays into particles. Gravitational perturbations affect the particle distribution, and to calculate it one has to specify the relative abundance of photons, neutrinos, baryons and cold dark matter before the second horizon crossing, to be used as an initial condition. Understanding perturbations in a multi-component medium is important both to analyse the anisotropy of the cosmic microwave background and to determine the relation between the primordial spectrum of density inhomogeneities created during inflation to the spectrum today. In general, perturbations of this system are of entropy or adiabatic types. Standard inflation assumes adiabatic fluctuations, where the entropy per baryon is conserved, and then a relation of the type $\delta\rho_r/\rho_r = (4/3)\delta\rho_m/\rho_m$ holds for a two-component fluid, where ρ_r and ρ_m are the energy density of radiation and non-relativistic matter, respectively.

Now, let us briefly mention tensor perturbations. All the process done for the scalar perturbation fields can be also done for tensor perturbations $h_{ij}(\vec{x},\eta)$ (see Dodelson (2003)). Because of (2.14), h_{ij} has 2 degrees of freedoms, or polarisations, which are usually indicated as $p=+,\times$. More precisely,

$$h_{ij}(\vec{x},\eta)=\int\frac{d^3k}{(2\pi)^{3/2}}h_{ij}(k,\eta)e^{i\vec{k}\cdot\vec{x}},\quad h_{ij}(k,\eta)=h_+(k,\eta)e^+_{ij}(k)+h_\times(k,\eta)e^\times_{ij}(k),\quad (2.96)$$

where e^+_{ij} and $e^\times_{ij}$ are the polarisation tensors, which have the following properties:

$$\begin{aligned}e^p_{ij}=e^p_{ji},\quad k^ie^p_{ij}=0,\quad e^p_{ii}=0,\\ e^p_{ij}(k)=e^{p*}_{ij}(-k),\quad e^{p*}_{ij}(k)e^{p'}_{ij}(k)=2\delta_{pp'}.\end{aligned}\quad (2.97)$$

Notice that h_{ij} is gauge-invariant and therefore represents a physical degree of freedom. A nearly scale-free power spectrum is also found when perturbations re-enter the horizon:

$$\mathcal{P}_\mathrm{T}=\frac{4k^3}{2\pi^2}|h|^2=\frac{2H_*^2}{\pi^2}\left(\frac{k}{a_*H_*}\right)^{n_\mathrm{T}},\quad (2.98)$$

where $h=h_\times=h_+$ and n_T is the spectral index for these tensor perturbations, whose value is:

$$n_\mathrm{T}=\left.\frac{d\ln(\mathcal{P}_\mathrm{T})}{d\ln(k)}\right|_{k=a_*H_*}=-2\epsilon_V.\quad (2.99)$$

These tensor perturbations generate primordial gravitational waves. So, inflation determines not just one but two primordial spectra, corresponding to the scalar and tensor perturbations. During inflation, no vector perturbations are generated.

Summarising, inflation predicts a nearly flat, homogeneous and isotropic Universe with small primordial perturbations. The power spectrum for the scalar perturbation at the second-horizon crossing, represented with $\mathcal{R}$, is nearly scale-invariant, and its deviation is given by the spectral index n_s, which is near to 1. Also, fluctuations are assumed to be Gaussian and adiabatic. These predictions are in accordance with observations of CMB anisotropies and the galaxy distribution today (see Komatsu et al. (2011)).

In this section, we have calculated a second-order action for the only physical scalar field present in the theory of inflation. We obtained a very simple action by fixing a convenient gauge and eliminating all non-physical degrees of freedom. We have quantised this field through the standard canonical quantisation process used in QFT. We discussed the problem related to the vacuum choice and used the Bunch-Davis vacuum, a state with no particles in the asymptotic past. We calculated the spectrum of the curvature perturbation field in the de Sitter background, but we finally extended it to inflation. We showed the nearly scale-invariant form of this spectrum, and mentioned the Gaussian and adiabatic characteristics of the predicted fluctuations. We finally mentioned the analogous result for tensor perturbations.

Chapter 3

Cosmological quantum perturbations in the Eddington-Born-Infeld theory

In this chapter, we will show the same calculations done in the previous chapter, now with the Eddington-Born-Infeld theory of gravity (EBI). We will first introduce the EBI theory, explaining its motivation and showing its action along with the equations of motion. We will see that this theory introduces modifications to General Relativity for large curvatures. Then, we will study the cosmological background, a homogeneous and isotropic Universe coupled to a perfect fluid. We will show the behaviour of the scale factor during the early Universe, i.e. in the radiation-dominated era. Later, we will consider linear perturbations on this background and write a second-order action for the physical tensor and scalar perturbation fields. Finally, we will quantise these fields and calculate their power spectra, using the method described in Hollands and Wald (2008). Throughout this chapter we will work with $8\pi G = 1$, $c = 1$, $\hbar = 1$ and signature (+,-,-,-).

3.1 Motivating the Eddington-Born-Infeld gravity

There are different motivations to study modifications of the Einstein-Hilbert action. One of them is the presence of singularities. Singularities have always been seen as shortcomings, because they signify a breakdown of standard concepts. For that reason, they are usually considered a sign of a missing piece in the theory. A very important singularity is predicted in standard cosmology: the Big Bang. This singularity is physical, and represents the birth of our Universe as a spacetime. A prevailing opinion on this phenomenon is that General Relativity breaks down at the Planck length scale, and a quantum gravity theory is appropriate to describe it. However, since there is no such quantum theory now, General Relativity can be modified at a classical level to address this issue. This is the spirit of EBI theory.

The EBI action (Bañados and Ferreira (2010)) was originally a modification of Eddington's action of gravity (Eddington (1924)) using the idea of Born-Infeld's action (Born and Infled (1934)) for electromagnetism, which we will show briefly.

Born-Infeld theory of electromagnetism

The Born-Infeld action is a modification of Maxwell's theory in order to avoid point-charge related divergences. It is a nonlinear theory defined by

$$S_{\mathrm{BI}}[g,F] = \frac{1}{b^2}\int d^4x \left(\sqrt{-g} - \sqrt{|g_{\mu\nu} + bF_{\mu\nu}|}\right), \tag{3.1}$$

where b is a constant, $F_{\mu\nu}$ the electromagnetic tensor and $g_{\mu\nu}$ the spacetime metric. Here, $|g_{\mu\nu}+bF_{\mu\nu}|$ means the absolute value of the determinant of $g_{\mu\nu}+bF_{\mu\nu}$. It can be observed that for a static point-charge, it predicts a finite electric field and energy everywhere.

If we build the Taylor series for small $bF_{\mu\nu}$ up to second order, we will obtain

$$S_{\mathrm{BI}}[g,F] \approx \frac{1}{4}\int d^4x \left(F^{\mu\nu}F_{\mu\nu}\right). \tag{3.2}$$

This means that the Born-Infeld's action reproduces the Maxwell theory for weak electromagnetic fields, but it introduces modifications in regions with strong fields.

Eddington's Action of Gravity

As it is known, General Relativity can be described by different types of actions. One of them is Eddington's action. It is a purely affine gravitational action, defined by:

$$S_{\mathrm{Edd}}[\Gamma] = -\frac{2}{\Lambda}\int d^4x \sqrt{|R_{\mu\nu}(\Gamma)|}, \tag{3.3}$$

which corresponds to the square root of the absolute value of the determinant of $R_{\mu\nu}$, the symmetric part of the Ricci tensor built solely from the affine connection $\Gamma^{\alpha}_{\beta\gamma}$. Here, the connection is assumed to be symmetric in the two low indices. This action is completely equivalent to the Einstein-Hilbert action with a cosmological constant. To see this we derive the equation of motion for Γ:

$$\nabla_{\alpha}\left(\frac{1}{\Lambda}\sqrt{|R|}R^{\mu\nu}\right)=0, \tag{3.4}$$

where the covariant derivative is defined in terms of the connection. We can define now a 2-rank tensor:

$$g_{\mu\nu}\equiv-\frac{1}{\Lambda}R_{\mu\nu}. \tag{3.5}$$

Then, the equation of motion turns out to be

$$\nabla_{\alpha}\left(\sqrt{|g|}g^{\mu\nu}\right)=0, \tag{3.6}$$

which implies that

$$\nabla_{\alpha}\left(g^{\mu\nu}\right)=0. \tag{3.7}$$

This equation represents a relation between $g^{\mu\nu}$ and the connection. In fact, this is the relation satisfied between a metric and its Christoffel symbols. Then, the connection Γ is the Christoffel symbol of the metric $g^{\mu\nu}$. So, eq. (3.5) turns out to be the Einstein-Hilbert equation:

$$R_{\mu\nu}(g)=-\Lambda g_{\mu\nu}. \tag{3.8}$$

Eddington's action can be viewed as dual with the Einstein-Hilbert action, because the Einstein-Hilbert action is proportional to the cosmological constant Λ and Eddington's is inversely proportional to Λ.

The Eddington-Born-Infeld action of gravity

As an attempt to modify gravity in the same way Born and Infeld did with electromagnetism, the Eddington-Born-Infeld action appears. It was proposed originally by Vollick (see Vollick (2004)), and it is defined by:

$$S_{\mathrm{EBI}}[g,\Gamma]=\frac{1}{\kappa}\int d^4x\left[\sqrt{|g_{\mu\nu}-\kappa R_{\mu\nu}(\Gamma)|}-\sqrt{-g}\right], \tag{3.9}$$

where κ is a constant with dimensions of L^2, g the metric and Γ the affine connection. Since it depends on the metric and the connection, we say that this action is a modification of Eddington's action.

Even though this action looks like a clear analogy to Born-Infeld's action, it does not share the same characteristics. This action is completely equivalent to Einstein-Hilbert's action, without a cosmological constant, for all values of κR.

In order to complete this gravitational theory, it is necessary to add matter. A simple way to do this is proposed in Bañados and Ferreira (2010):

$$S_{\text{EBI}}[g, \Gamma, \chi] = \frac{1}{\kappa} \int d^4x \left[\sqrt{|g_{\mu\nu} - \kappa R_{\mu\nu}(\Gamma)|} - \lambda\sqrt{-g} \right] + S_{\text{m}}[\chi, g], \tag{3.10}$$

where S_m is the matter action which depends on a field χ, and λ is a non-zero constant related to the cosmological constant:

$$\Lambda = \frac{\lambda - 1}{\kappa}. \tag{3.11}$$

Then, for $\lambda = 1$, as in (3.9), this action describes a theory without a cosmological constant.

Now that matter has been added, this action is only equivalent to Einstein-Hilbert's action in a weak-field limit. This can be seen by building the Taylor series for small κR up to first order, where we would obtain:

$$S_{\text{EBI}}[g, \Gamma, \chi] \approx -\frac{1}{2} \int d^4x\sqrt{-g} \left[g^{\mu\nu} R_{\mu\nu}(\Gamma) + 2\left(\frac{\lambda - 1}{\kappa}\right) \right] + S_{\text{m}}[\chi, g], \tag{3.12}$$

which is a metric-affine action equivalent to Einstein-Hilbert's action. As we will confirm later, as a result of adding matter, this action actually modifies General Relativity in regions with strong fields, and it results in a complete analogy to Born-Infeld's action for electromagnetism.

In this section we explained the motivation to develop the EBI theory. We showed the explicit form of this action, along with the two theories it is based on. We mentioned that the EBI action reproduces General Relativity in the weak-field limit, but it introduces modifications in regions with high curvature.

Next, we will study the EBI theory in more detail.

3.2 Eddington-Born-Infeld Action

In this section we will study the EBI theory, its equations of motion and the fact that it reproduces General Relativity, with more details.

Form now on, we will consider the Eddington-Born-Infeld action with matter written as

$$S[q,g,\chi] = -\frac{1}{2}\int d^4x \left[\sqrt{-q}\left(R(q)+\frac{2}{\kappa}\right) - \frac{1}{\kappa}\left(\sqrt{-q}q^{\mu\nu}g_{\mu\nu} - 2\sqrt{-g}\right)\right] + S_{\rm m}[\chi, g]. \quad (3.13)$$

This action depends on two metrics: g and q, plus a matter field χ. We notice that g is the physical metric due to the fact it is coupled to matter. This action is completely equivalent to (3.10) with $\lambda = 1$ (see appendix A).

The equations of motion of this theory are

$$q_{\mu\nu} = g_{\mu\nu} - \kappa R_{\mu\nu}(q), \quad (3.14a)$$

$$\sqrt{-q}q^{\mu\nu} = \sqrt{-g}g^{\mu\nu} + \kappa T^{\mu\nu}, \quad (3.14b)$$

where we have used that

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g_{\mu\nu}}. \quad (3.15)$$

Also, we have to consider the conservation law for matter:

$$T^{\mu\nu}{}_{;\nu} = 0. \quad (3.16)$$

Now that we have got the equations of motion, we can study some characteristics of this theory. From (3.14), we can see that if $T^{\mu\nu} = 0$:

$$g_{\mu\nu} = q_{\mu\nu} \quad \Rightarrow \quad R_{\mu\nu}(g) = 0. \quad (3.17)$$

Thus, in the absence of matter, action (3.13) reproduces General Relativity in vacuum. This is one of the characteristics we mentioned in the previous section for action (3.9). Moreover, if κR is small, action (3.13) reproduces the Einstein-Hilbert action. Let us see this with the equations of motion. Take (3.14a) to calculate the determinant of q and build the Taylor series up to first order in κR:

$$q_{\mu\nu} = g_{\mu\nu} - \kappa R_{\mu\nu}(q) \quad \Rightarrow \quad \sqrt{-q} = \sqrt{-g} - \frac{1}{2}\sqrt{-g}g^{\mu\nu}\kappa R_{\mu\nu}(g) + \mathcal{O}\left((\kappa R_{\mu\nu})^2\right)$$
$$\Rightarrow \quad q^{\mu\nu} = g^{\mu\nu} + \kappa g^{\mu\alpha}R_{\alpha\beta}(g)g^{\beta\nu}. \tag{3.18}$$

Replacing this into (3.14b), we obtain:

$$\left(\sqrt{-g} - \frac{1}{2}\sqrt{-g}\kappa R(g)\right)\left(g^{\mu\nu} + \kappa R^{\mu\nu}(g)\right) = \sqrt{-g}g^{\mu\nu} + \kappa T^{\mu\nu}$$
$$\Rightarrow \quad R^{\mu\nu}(g) - \frac{1}{2}R(g)g^{\mu\nu} = T^{\mu\nu}, \tag{3.19}$$

which is the equation of motion of the Einstein-Hilbert action coupled to matter with a stress-energy tensor given by $T^{\mu\nu}$.

We conclude that the EBI theory of gravity reproduces General Relativity in the weak field limit, and they are exactly the same in the absence of matter.

Next, we will study the EBI theory in the context of cosmology, in the same way as we did in the previous chapter with General Relativity. We will see that modifications of General Relativity for regions with strong fields bring a relevant consequence during the early Universe.

3.3 Cosmological background and perturbations

In this section, we will study the equations of motion in the EBI theory for a homogeneous, flat and isotropic background Universe in presence of a perfect fluid. Also, we will consider linear perturbations for both metrics and matter fields in the background studied.

3.3.1 Background: homogeneous and isotropic Universe

In the EBI theory, we can study the solution for a homogeneous and isotropic Universe coupled to a perfect fluid, with the following stress-energy tensor:

$$T^{\mu\nu} = (\rho_0 + p_0)u_0^\mu u_0^\nu - p_0 g^{(0)\mu\nu}, \tag{3.20}$$

where the super index (0) for the metric and the sub index 0 for the rest energy density, pressure and cuadrivelocity denote a background value. This background was studied in Bañados and Ferreira (2010). It was found that the line element solution for both metrics can be put into the following form:

$$ds_g^{(0)2} = a(\eta)^2[d\eta^2 - d\vec{x}\cdot d\vec{x}], \tag{3.21a}$$
$$ds_q^{(0)2} = b(\eta)^2[z(\eta)^{-1}d\eta^2 - d\vec{x}\cdot d\vec{x}], \tag{3.21b}$$

where $a(\eta)$, $b(\eta)$ and $z(\eta)$ are functions determined by the equations of motion, and η is the conformal time. Notice that the physical metric g has the standard form for a homogeneous and isotropic expanding flat Universe, with $a(\eta)$ being the scale factor. For simplicity, from now on, we will omit the dependence on η.

Replacing (3.21) into the equations of motion (3.14) and (3.16), given in the previous section, we find the following relevant equations for this background:

$$\sqrt{z}b^2 - a^2(\kappa\rho_0 + 1)) = 0, \tag{3.22}$$
$$a^2(\kappa p_0 - 1)\sqrt{z} + b^2 = 0, \tag{3.23}$$
$$a^2 z b^2 - 6z\kappa b'^2 + 2b^4 - 3a^2b^2 = 0, \tag{3.24}$$
$$\rho_0' + 3\left(\rho_0 + p_0\right)\mathcal{H} = 0, \tag{3.25}$$

where $\mathcal{H} \equiv a'/a$ is the comoving Hubble parameter. By combining these equations we obtain:

$$z = (1 + \kappa\rho_0)/(1 - \kappa p_0), \tag{3.26}$$
$$b = a[(1 - \kappa p_0)(1 + \kappa\rho_0)]^{1/4}. \tag{3.27}$$

Using all these equations of motion we find the Friedmann equation with $p_0 = \omega\rho_0$:

$$\mathcal{H}^2 = 3\frac{a^2}{\kappa}(1 + \kappa\rho_0)(1 - \kappa\rho_0\omega)^2 \frac{\left[\frac{1}{2}(1 + 3\omega)\kappa\rho_0 - 1\right] + \sqrt{(1 + \kappa\rho_0)(1 - \kappa\rho_0\omega)^3}}{\left[3 + \frac{3}{2}\omega(1 + 3\omega)\kappa^2\rho_0^2 + \frac{3}{4}(3\omega - 1)(\omega - 1)\kappa\rho_0\right]^2}. \tag{3.28}$$

Let us study this equation in different cases:

- Late times: $\omega = -1$, i.e. a Universe dominated by a cosmological constant. In this case:

$$\mathcal{H}^2 = \frac{1}{3}a^2\rho_0, \tag{3.29}$$

which is the same Friedmann equation given by General Relativity.

- Small densities: $\kappa\rho_0 \ll 1$. In this case,

$$\mathcal{H}^2 = a^2\frac{1}{3}\rho_0, \tag{3.30}$$

which is the one obtained in standard General Relativity, as we expected.

- Early times: $\omega = 1/3$, i.e. a Universe dominated by radiation. In this case, the Friedmann equation is

$$\mathcal{H}^2 = a^2\frac{(\kappa\rho_0+1)(3-\kappa\rho_0)^2}{3\kappa(3+\kappa^2\rho_0^2)^2}\left(\frac{1}{3\sqrt{3}}\sqrt{[(3-\kappa\rho_0)]^3(\kappa\rho_0+1)}+\kappa\rho_0-1\right), \tag{3.31}$$

which is different to that from General Relativity. Then, we expect to have an evolution of the scale factor different to General Relativity at early times, but similar at late times.

Since we are interested in calculating primordial perturbations, we will focus our attention on the characteristics of this theory at early times. From (3.31), we can see two stationary points where $\mathcal{H}(\eta) = 0$: at $\rho_0 = 3/\kappa$ for $\kappa > 0$, and at $\rho_0 = -1/\kappa$ for $\kappa < 0$. This means there is a minimum value a_B of the scale factor which, according to (3.25), corresponds also to a maximum value ρ_B of the energy density. The value of ρ_B is $-1/\kappa$ or $3/\kappa$, depending on the sign of κ. The minimum value a_B can be expressed in terms of ρ_B as $a_B = (\rho_T/\rho_B)^{-4}$, where ρ_T is the energy density of radiation today. Notice that the value a_B is different for the two signs of κ.

At these early times, the behaviour of the scale factor as a function of η near the minimum of the scale factor a_B is:

$$\kappa > 0: \quad a(\eta) = a_B\left(e^{\sqrt{\frac{8}{3\kappa}}a_B(\eta-\eta_0)}+1\right), \tag{3.32}$$

$$\kappa < 0: \quad a(\eta) = a_B\left(1+\frac{2a_B^2}{3|\kappa|}(\eta-\eta_0)^2\right). \tag{3.33}$$

Figure 3.1 shows the evolution of the scale factor as a function of the physical time t for both cases: $\kappa > 0$ and $\kappa < 0$, in the radiation-dominated era. When $\kappa > 0$, the early Universe initially grows exponentially fast with time (see (3.32)). When this is happening we say we are in the Eddington period. The minimum of the scale factor a_B happens when $t \to -\infty$. As time goes on, the energy density is getting smaller, and the scale factor evolves as predicted in General Relativity, so we say we are in the Einstein period. On the other hand, when $\kappa < 0$, the evolution of $a(t)$ is different during the early Universe: the scale factor diminishes until a bounce occurs. This is in the Eddington period. Consequently, the minimum a_B is reached in a finite time, after which the scale factor evolves as it does in General Relativity, in the Einstein period. We recall that the motivation of the EBI theory was to eliminate divergences, and here we can see that in both cases the Big Bang divergence disappears indeed.

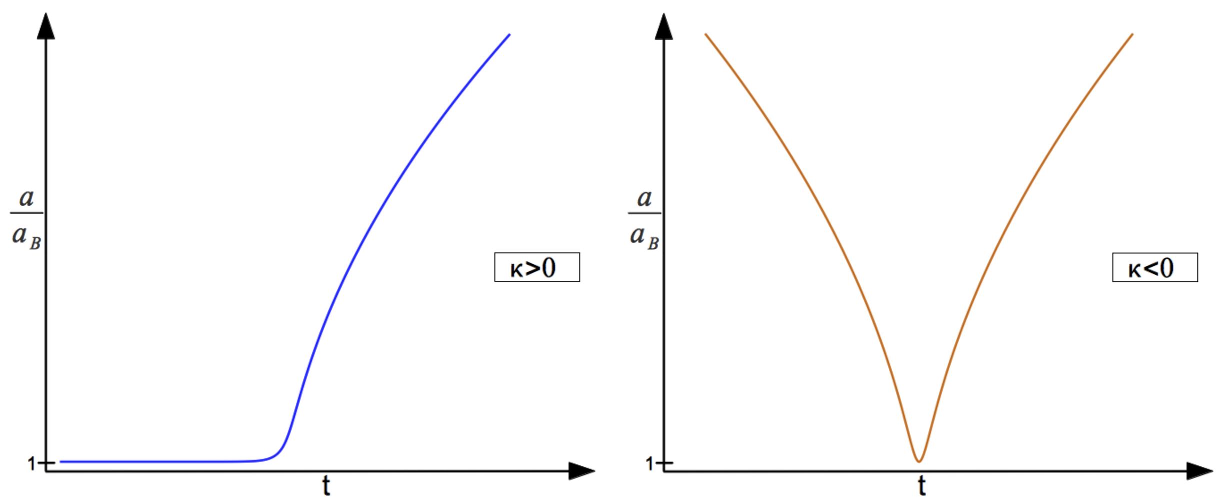

Figure 3.1: Scale factor as a function of physical time t. A minimum value a_B is found in both cases. For $\kappa > 0$, the early Universe is initially expanding exponentially fast (blue line), with $a \to a_B$ while $t \to -\infty$. For $\kappa < 0$, there is a bounce (gold line), and the minimum value a_B occurs in a finite time. There is no presence of Big Bang in any case.

From (3.32)-(3.33), we can see that in both cases $\ddot{a} > 0$ near a_B. Therefore, there is a period of inflation (accelerated expansion) during the early Universe. Then, the horizon problem is solved naturally in this theory. There is no need to include any fundamental scalar field with a special condition in the state equation, as we did in the previous chapter for the inflation theory.

Figure 3.2 shows the evolution of the comoving Hubble radius (blue and gold lines) as a function of the scale factor a (in analogy to Fig. 2.1) during the radiation-dominated era, for both cases of κ. In the case of $\kappa > 0$, the comoving Hubble radius initially behaves as $R_c = \sqrt{\frac{3\kappa}{8a_B}\frac{1}{(a-a_B)}}$, but later it behaves as it is predicted by General Relativity, i.e. grows linearly with a. On the other hand, for $\kappa < 0$, it initially behaves as $R_c = \sqrt{\frac{3|\kappa|}{8a_B}\frac{1}{\sqrt{a-a_B}}}$, and later as predicted by General Relativity. This figure also shows a given comoving scale λ (green line), which first is a sub-Hubble scale, and thus, within a region of size λ, physical information is being transmitted. It then becomes a super-Hubble scale, and then again a sub-Hubble scale.

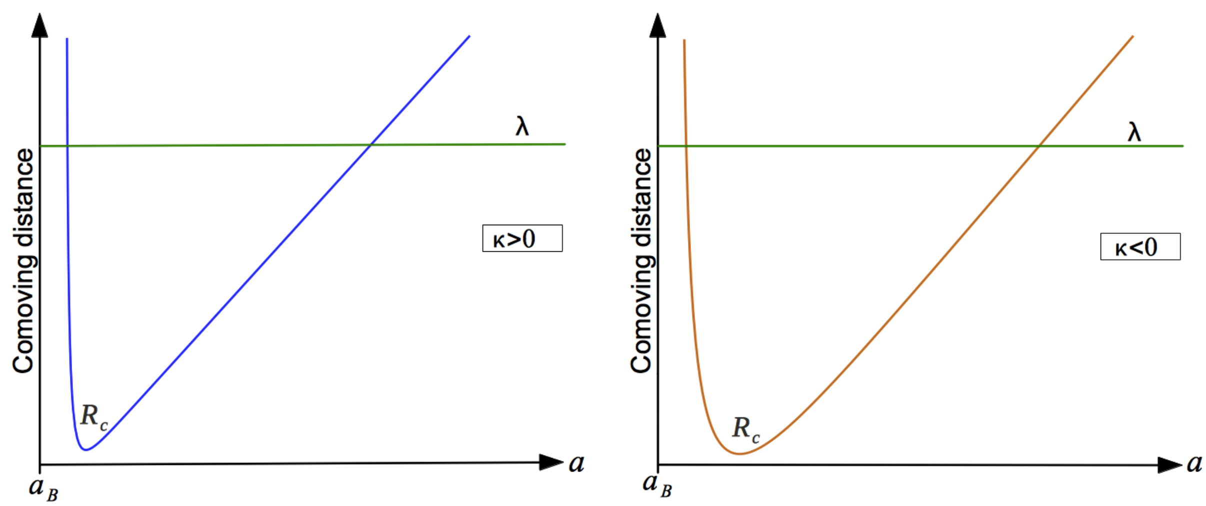

Figure 3.2: Evolution of the comoving Hubble radius R_c (blue and gold lines) as a function of the scale factor a, in a radiation-dominated Universe. There is initially a contracting period of R_c, where it evolves differently in the two cases of κ. Later, R_c grows linearly, as predicted by General Relativity.

In both cases, the expansion of R_c is infinitely large as we approach a_B. In particular, in the early Universe, R_c was bigger than the size of the comoving horizon at the time of the photon decoupling or earlier, and consequently there is no horizon problem.

On the other hand, to study the flatness problem we look at the Friedmann equation for a spatially non-flat Universe during the radiation-dominated era:

$$\mathcal{H}^2 = a^2 \frac{(\kappa\rho_0 + 1)(3 - \kappa\rho_0)^2}{3\kappa(3 + \kappa^2\rho_0^2)^2} \left(\frac{1}{3\sqrt{3}} \sqrt{[(3 - \kappa\rho_0)]^3(\kappa\rho_0 + 1)} + \kappa\rho_0 - 1 + \frac{k\kappa}{a^2}(\kappa\rho_0 - 3) \right), \tag{3.34}$$

where k is a constant parameter related to the curvature of the space. Let us recall that a critical density ρ_c is defined as the value of ρ_0 for a spatially flat Universe. Notice that when $\mathcal{H} = 0$, we have $\rho_0 = \rho_c = \rho_B$, consequently we have $\Omega = \frac{\rho_0}{\rho_c} = 1$. Then, the equations in this theory give naturally a time when the Universe has a critical energy density ρ_c, and therefore they set the initial condition $\Omega = 1$ at $a = a_B$. At least for $\kappa > 0$, this initial condition is intuitive because as time goes on Ω always diverges from 1, and consequently as we move backwards, after infinite time, Ω will approaches the value 1 asymptotically. In addition, since during the Eddington period the scale factor does not grow much, Ω will never deviate much from 1, making it natural to have Ω very near 1 at the beginning of the Einstein period. This solves the flatness problem. For $\kappa < 0$, the condition $\Omega = 1$ at a finite time (when $a = a_B$) does not seem natural because there is no reason to expect $\Omega = 1$ at $a = a_B$ more than any other value. However, some authors affirm it solves the flatness problem (see Avelino and Ferreira (2012)).

As we mentioned in the previous chapter, the theory of inflation is needed to describe the early Universe well in General Relativity. The initial motivations were the horizon and the flatness problems. In this subsection we saw that these problems do not arise in the EBI theory. It seems, then, that this theory could be a good alternative to inflation, where no unknown scalar field is forced to be considered. But, for this to be true, the EBI theory needs to reproduce the major predictions of inflation: primordial nearly scale-invariant spectra for scalar and tensor perturbations at the second horizon-crossing. In order to see whether these predictions for fluctuations are the same in this theory, we will study the linear theory of perturbations.

3.3.2 Theory of linear perturbations

As in chapter 2, we will assume that primordial quantum perturbations were originated during the early Universe. Because of gravitational instability, these perturbations grew until structures were formed through gravitational collapse. An important difference is that in the previous chapter, the early Universe was dominated by the energy of a scalar field, but in the EBI theory, the early Universe is the standard radiation-dominated era. For this reason, from now on, we will use $\omega = 1/3$.

We will now study the linear perturbation theory for the EBI action, with the background described above. We will follow the same classification for metric perturbations as in section 2.1.3. Since the three types of perturbations are independent, we will study each one separately.

In order to describe perturbations of a general perfect fluid, we will follow the idea presented in Mukanov, Feldman and Brandenberger (1992). We can think of the fluid as formed by many particles with rest mass m_0. The perturbation fluid is then described by a shift vector χ^μ which shifts the position of a test particle from x_0^μ, where it would be in an unperturbed Universe. Notice that we can decompose the fluid shift vector into the transverse and longitudinal parts:

$$\chi^\mu = (\chi^0, \chi^i), \quad \text{where} \quad \chi^i = \chi^{iT} + \partial_i \chi, \quad \text{such that} \quad \partial_i \chi^{iT} = 0. \tag{3.35}$$

Here, we shift the three spatial indices by using the metric δ_{ij} and its inverse δ^{ij}. We can see that the scalar part of the shift vector is χ and χ^0, which will couple to scalar metric perturbations. The vector part is χ^{iT}, which will couple to vector metric perturbations. And there is no tensor part in χ^μ, so tensor metric perturbations do not couple to matter perturbations. Notice that we will be using the equation of state $p = \frac{1}{3}\rho$, which is satisfied for zero and first order perturbations.

Scalar perturbations

Consider scalar perturbations for both metrics on the background described above:

$$ds_q^2 = b^2[z^{-1}(1+2\phi_1)d\eta^2 - 2B_{1,i}\sqrt{z}^{-1}dx^i d\eta - [(1-2\psi_1)\delta_{ij} + 2E_{1,ij}]dx^i dx^j], \tag{3.36}$$
$$ds_g^2 = a^2[(1+2\phi_2)d\eta^2 - 2B_{2,i}dx^i d\eta - [(1-2\psi_2)\delta_{ij} + 2E_{2,ij}]dx^i dx^j]. \tag{3.37}$$

We read from here that the scalar perturbation fields for both metrics are ϕ_1, B_1, E_1, ψ_1 for q and ϕ_2, B_2, E_2, ψ_2 for g.

On the other hand, these scalar metric fields are coupled to matter perturbations. If we write the stress-energy tensor as (2.10) and consider only the scalar matter perturbations, the stress-energy tensor perturbation will be

$$\begin{aligned}
\delta T^0{}_0 =&(\rho_0+p_0)(3\psi_2-E_{2,ii}-\chi_{,ii}),\\
\delta T^i{}_0 =&(\rho_0+p_0)\chi'_{,i},\\
\delta T^0{}_i =&-(\rho_0+p_0)(B_{2,i}+\chi'_{,i}),\\
\delta T^i{}_j =&-\frac{1}{3}(\rho_0+p_0)(3\psi_2-E_{2,ll}-\chi_{,ll})\delta^i{}_j.
\end{aligned} \tag{3.38}$$

From here we can see that χ^0 does not appear, then χ is the only scalar matter degree of freedom. For more information about matter perturbations and this stress-energy tensor perturbation, see Appendix B.

Finally, we have 9 scalar perturbation fields in the EBI theory with a perfect fluid: 4 for each metric plus 1 for matter.

Vector perturbations

We consider vector perturbations for both metrics:

$$ds_q^2 = b^2[z^{-1}d\eta^2 + 2S_{1i}\sqrt{z}^{-1}dx^i d\eta - (\delta_{ij}+F_{1i,j}+F_{1j,i})dx^i dx^j], \tag{3.39}$$

$$ds_g^2 = a^2[d\eta^2 + 2S_{2i}dx^i d\eta - (\delta_{ij}+F_{2i,j}+F_{2j,i})dx^i dx^j]. \tag{3.40}$$

From here we can see that the vector perturbations are S_{1i} and F_{1i} for the metric q, and S_{2i} and F_{2i} for g. Each vector perturbation satisfies (2.12). We recall that we shift from upper to lower spatial indices and vice versa by using the metric δ_{ij} and its inverse δ^{ij}.

If we consider only the vector matter perturbations, the stress-energy tensor perturbation is:

$$\begin{aligned}
\delta T^0{}_0 &= 0,\\
\delta T^i{}_0 &= (\rho_0+p_0)\chi^{iT'},\\
\delta T^0{}_i &= -(\rho_0+p_0)(\chi^{iT'}-S_{2i}),\\
\delta T^i{}_j &= 0,
\end{aligned} \tag{3.41}$$

where $v^{iT}\equiv\chi^{iT'}$ represents the vorticity of the fluid and satisfies $v^{iT}{}_{,i}=0$.

Finally, we have 5 vector perturbation fields: 2 for each metric plus 1 of matter.

As we said in the previous chapter, vector perturbations have no cosmological relevance in General Relativity with a perfect fluid. This theory shows the same behaviour. To see this, let us write the equations of motion for the perturbation fields in the Fourier space with a fixed gauge.

The EBI action is invariant under the following vector transformations:

$$\begin{aligned}
&\tilde{F}_{2i} = F_{2i} - \xi^{iT}, \quad \tilde{S}_{2i} = S_{2i} + \xi^{iT'}, \quad \tilde{v}^{iT} = v^{iT} + \xi^{iT'}, \\
&\tilde{F}_{1i} = F_{1i} - \xi^{iT}, \quad \tilde{S}_{1i} = S_{1i} + \xi^{iT'},
\end{aligned} \tag{3.42}$$

where ξ^{iT} is an infinitesimal arbitrary vector field, satisfying $\xi^{iT}{}_{,i} = 0$.

With the gauge choice $\tilde{F}_{1i} = 0$, the relevant equations of motion are:

$$F_{2i} = 0, \tag{3.43}$$

$$S_{2i} - \sqrt{z} S_{1i} + (z-1) v^{iT} = 0, \tag{3.44}$$

$$2 S_{1i} a^2 - 2a^2 S_{2i} \sqrt{z} - S_{1i} \kappa k^2 = 0, \tag{3.45}$$

$$2a^2 F_{2i} \sqrt{z} + \kappa z (2 S_{1i} h + S'_{1i}) = 0. \tag{3.46}$$

Using the first 3 equations we can write all the fields in terms of the vorticity field v^{iT}, and obtain:

$$\frac{dv^{iT}}{d\eta} + \frac{2\left[h(\kappa k^2 + 2a^2(z-1)) + \frac{z'}{4z}\left(2(z-1)a^2 + \kappa k^2 \frac{(3z-1)}{(z-1)}\right) + \kappa k^2 \mathcal{H}\right]}{(z-1)(\kappa k^2 + 2a^2(z-1))} v^{iT} = 0. \tag{3.47}$$

The solution of this equation is $v^{iT} \approx$ constant during the radiation-dominated era and $v^{iT} \propto 1/a$ during matter-dominated era. As a consequence, the vector perturbation of the physical metric is $S_{2i} \approx$ constant during the Eddington period (early radiation-dominated era) and $S_{2i} \propto 1/a^2$ during the Einstein period of radiation and matter. Since all these perturbations decay, primordial vector fluctuations would have a significant amplitude at present only if they were originally very large. There is no reason to expect such large primordial fluctuations, so from now on we will completely ignore them, and focus only on scalar and tensor perturbations.

Tensor perturbations

We consider tensor perturbations for both metrics:

$$ds_q^2 = b^2[z^{-1}d\eta^2 - (\delta_{ij} + h_{1ij})dx^i dx^j], \tag{3.48}$$

$$ds_g^2 = a^2[d\eta^2 - (\delta_{ij} + h_{2ij})dx^i dx^j]. \tag{3.49}$$

From here we can see that the tensor perturbations are h_{1ij} for the metric q, and h_{2ij} for g. These perturbations satisfy (2.14). Again, we use the metric δ_{ij} and its inverse δ^{ij} to shift spatial indices. Since in this theory there are no tensor matter perturbations, the perturbed stress-energy tensor to be considered here is zero.

To sum up, in this section we calculated the main equations governing the evolution of a homogeneous and isotropic Universe in expansion, in the EBI theory. As expected, we found at early times a different behaviour to that of General Relativity. During the radiation era, we saw two different evolutions of the Universe, depending on the sign of κ: one where the scale factor grows exponentially fast, and another where there is a bounce. In both cases, the Universe does not have a beginning nor an end; there is a minimum value for the scale factor and a maximum energy density. So, there is no Big Bang divergence. Also, in both cases there is an early period of inflation (accelerated expansion), which allows us to avoid the horizon problem. In addition, the $\kappa > 0$ case solves the flatness problem. We started the linear perturbation theory in the homogeneous and isotropic background. We showed both metrics and stress-energy tensor perturbations during the radiation-dominated era explicitly. We showed that vector perturbations have no cosmological relevance, which allows us to ignore them.

Having all the perturbation fields written, the next step is to proceed with the quantum theory. In order to do that, we must calculate an action for all the physical degrees of freedom, and quantise them. However, we must first discuss the gauge-fixing.

3.4 Gauge invariance

The EBI theory has a symmetry that comes from considering general coordinate transformations. However, in the linear perturbation theory, this action is invariant only under infinitesimal transformations up to first order.

Let us consider infinitesimal coordinate transformations. Suppose we perform the following change of coordinates:

$$\tilde{x}^\mu = x^\mu + \xi^\mu, \tag{3.50}$$

where $\xi^\mu = \xi^\mu(x)$ is an arbitrary (infinitesimal) vector function. We split ξ^μ,

$$\xi^\mu = (\xi^0, \xi^i), \quad \text{where} \quad \xi^i = \xi^{iT} + \partial_i \xi \quad \text{and} \quad \partial_i \xi^{iT} = 0. \tag{3.51}$$

Again, the spatial index i is moved with the euclidean metric. Since tensor, vector and scalar parts are decoupled, ξ^0 and ξ are related to the scalar perturbations and ξ^{iT} is related to vector perturbations. Since there is no tensor gauge freedom, fields h_{1ij} and h_{2ij} are gauge-invariant. Therefore, during this section we will focus only on scalar perturbations.

Under the infinitesimal coordinate transformation (3.50), the scalar perturbation fields described previously in the cosmological background will transform to tilde fields as (see Mukanov, Feldman and Brandenberger (1992)):

$$\begin{aligned}
&\tilde{\phi}_2 = \phi_2 - \frac{a'}{a}\xi^0 - \xi^{0\prime}, \;\; \tilde{\psi}_2 = \psi_2 + \frac{a'}{a}\xi^0, \;\; \tilde{B}_2 = B_2 + \xi^0 - \xi', \;\; \tilde{E}_2 = E_2 - \xi, \\
&\tilde{\phi}_1 = \phi_1 - \left[\frac{b'}{b} - \frac{z'}{2z}\right]\xi^0 - \xi^{0\prime}, \;\; \tilde{\psi}_1 = \psi_1 + \frac{b'}{b}\xi^0, \;\; \tilde{B}_1 = B_1 - \xi'\sqrt{z} + \frac{\xi^0}{\sqrt{z}}, \;\; \tilde{E}_1 = E_1 - \xi, \\
&\tilde{\chi} = \chi + \xi.
\end{aligned} \tag{3.52}$$

As we explained in section 2.3, these gauge freedoms can cause some confusions when trying to calculate physical quantities. To avoid this problem, we will follow the same idea as in inflation, i.e. use a manifestly gauge-invariant framework. Specifically, we are going to fix all the gauge freedoms and eliminate non-physical perturbation fields, in such a way that the final physical degrees of freedom are gauge-invariant, as expected. This method is very useful. In fact, in the EBI theory we found 9 scalar perturbation fields, which apparently would lead to very long calculations. However, we will see that this number is easily reduced to 1 if we follow this method.

Next, we will show the gauge choice that will be used in the next sections to study quantum scalar perturbations.

3.4.1 Gauge choice

In section 2.3 we discussed the subtleties related to the gauge choice when one is working with the action, as we will do in the quantum theory of perturbations. We can only fix the value of fields whose equations of motion are redundant. For that reason, here we will follow the same procedure as in that section to make an appropriate gauge choice.

Let us consider a general variation of the EBI perturbed action (action of second order in the 9 scalar perturbation fields):

$$\delta S = \int \left(Eq_{\phi_1}\delta\phi_1 + Eq_{\psi_1}\delta\psi_1 + Eq_{E_1}\delta E_1 + Eq_{B_1}\delta B_1 + Eq_{\chi}\delta\chi \right. \tag{3.53}$$

$$\left. + Eq_{\phi_2}\delta\phi_2 + Eq_{\psi_2}\delta\psi_1 + Eq_{E_2}\delta E_2 + Eq_{B_2}\delta B_2 \right), \tag{3.54}$$

where Eq_n is the equation of motion for the field n. But if we replace these variations of fields δn by the gauge variations given in (3.52), we will obtain $\delta S = 0$, because the EBI action is gauge-invariant. Let us do this and perform some integration by parts:

$$\delta S_{\text{gauge}} = \int \Bigg(Eq'_{\phi_1} - Eq_{\phi_1}\left[\frac{b'}{b} - \frac{z'}{2z}\right] + Eq_{\psi_1}\frac{b'}{b} + \frac{Eq_{B_1}}{\sqrt{z}} + Eq'_{\phi_2} + \left(Eq_{\psi_2} - Eq_{\phi_2}\right)\frac{a'}{a} \\ + Eq_{B_2}\Bigg)\xi^0 + \Bigg(-Eq_{E_1} + (Eq_{B_1}\sqrt{z})' - Eq_{E_2} + Eq'_{B_2} + Eq_{\chi}\Bigg)\xi. \tag{3.55}$$

Since $\delta S_{\text{gauge}} = 0$, both parentheses must be zero because ξ and ξ^0 are completely arbitrary. This gives us two relations between the equations of motion:

$$Eq'_{\phi_1} - Eq_{\phi_1}\left[\frac{b'}{b} - \frac{z'}{2z}\right] + Eq_{\psi_1}\frac{b'}{b} + \frac{Eq_{B_1}}{\sqrt{z}} + Eq'_{\phi_2} + \left(Eq_{\psi_2} - Eq_{\phi_2}\right)\frac{a'}{a} + Eq_{B_2} = 0, \\ -Eq_{E_1} + (Eq_{B_1}\sqrt{z})' - Eq_{E_2} + Eq'_{B_2} + Eq_{\chi} = 0. \tag{3.56}$$

Any field that can be worked out from these two last equations has a redundant equation of motion. Then, the possibilities are:

$$(\psi_1, \psi_2) + (E_1, E_2, \chi), \tag{3.57}$$

Equation (3.57) means that we can set one field from the first parenthesis plus one field from the second parenthesis to zero. For convenience in later calculations, we will choose

$$\tilde{\psi}_1 = 0, \quad \tilde{\chi} = 0. \tag{3.58}$$

These conditions fix ξ and ξ^0 uniquely to be:

$$\xi = -\chi, \quad \xi^0 = -\frac{b'}{b}\psi_1. \tag{3.59}$$

We have managed to reduced our initial problem with 9 scalar perturbation fields to one with 7 fields. This gauge choice is good because if we define the following gauge-invariant field:

$$\zeta \equiv \psi_2 - \frac{1}{3(\rho_0 + p_0)}\delta\rho, \tag{3.60}$$

where $\delta\rho$ is the first order energy density fluctuation given by the $\delta T^0{}_0$ component of (3.38), it turns out to be proportional to the perturbative metric variable E_1 in the Fourier space in our gauge choice:

$$\zeta = -\frac{1}{3}k^2 E_1. \tag{3.61}$$

This variable is called curvature perturbation on slices of uniform energy density and we will use it to represent the only physical scalar degree of freedom in this theory. We expect to have only one physical scalar degree of freedom because General Relativity without matter has none, and therefore the EBI theory without matter neither. As a consequence, all scalar degrees of freedom arise from the matter action, and we already saw that there is only one: χ.

In this section we studied the gauge symmetry of the EBI theory, showing explicitly how the scalar perturbation fields transform. We also found all the possibilities to fix the scalar gauge freedoms, and chose one for later use.

The next step is to study the quantum linear perturbation theory for scalar and tensor perturbations.

3.5 Quantum perturbation theory

Following the same ideas of section 2.4, we will first obtain an action to second order in all the perturbations. After that, we will calculate a reduced action for only the physical fields.

3.5.1 Second-order action

In this subsection we will find the perturbed second-order EBI action for all the scalar and tensor perturbation fields. This second-order action will give first order equations of motion for the perturbation fields. We will initially make calculations only for the gravitational part of the EBI action, and later for the matter part. In order to keep a general calculation, we will not fix the gauge freedoms in this subsection, but in the next one.

We split the gravitational part of the EBI action (3.13) into two parts:

$$S_1 = -\frac{1}{2}\int d^4x\sqrt{-q}\left(R(q) + \frac{2}{\kappa}\right), \tag{3.62}$$

and

$$S_2 = \frac{1}{2\kappa}\int d^4x(\sqrt{-q}q^{\mu\nu}g_{\mu\nu} - 2\sqrt{-g}). \tag{3.63}$$

Gravitational scalar action

We must replace in S_1 and S_2 all the scalar perturbation fields given in (3.36) and (3.37), and expand a Taylor series up to second order. For example, we have to expand the determinants:

$$\begin{aligned}
\sqrt{-q}^{(0)} &= \frac{b^4}{\sqrt{z}}, \\
\sqrt{-q}^{(1)} &= \frac{b^4}{\sqrt{z}}\left[\phi_1 - 3\psi_1 + E_{1,ii}\right], \\
\sqrt{-q}^{(2)} &= \frac{b^4}{\sqrt{z}}\left[-\frac{1}{2}\phi_1^2 - 3\phi_1\psi_1 + \frac{3}{2}\psi_1^2 + \frac{1}{2}B_{1,i}B_{1,i} - \frac{1}{2}E_{1,ii}E_{1,jj} - E_{1,ii}\psi_1 + E_{1,ii}\phi_1\right],
\end{aligned} \tag{3.64}$$

and

$$
\begin{aligned}
\sqrt{-g}^{(0)} &= a^4, \\
\sqrt{-g}^{(1)} &= a^4 \left[\phi_2 - 3\psi_2 + E_{2,ii}\right], \\
\sqrt{-g}^{(2)} &= a^4 \left[-\frac{1}{2}\phi_2^2 + \frac{1}{2}B_{2,i}B_{2,i} - 3\phi_2\psi_2 + \phi_2 E_{2,ii} + \frac{3}{2}\psi_2^2 - \frac{1}{2}E_{2,ii}E_{2,jj} - E_{2,ii}\psi_2\right],
\end{aligned}
\tag{3.65}
$$

where the super indices (0), (1) and (2), mean zero, first and second-order perturbations, respectively. The calculations for obtaining the second-order action are straightforward but long. Action S_1 up to second order is:

$$
\begin{aligned}
S_1^{(2)} = \frac{1}{2}\int d^4x \frac{b^2}{\sqrt{z}} \Big[& -6z\psi_1'^2 - 12zh(\phi_1+\psi_1)\psi_1' - 9zh^2(\phi_1+\psi_1)^2 - 2\psi_{1,i}(2\phi_{1,i} - \psi_{1,i}) \\
& - 4\sqrt{z}h(\phi_1+\psi_1)(B_1 - \sqrt{z}E_1')_{,ii} + 4zh\psi_1' E_{1,ii} - 4\sqrt{z}\psi_1'(B_1 - \sqrt{z}E_1')_{,ii} - 4h\psi_{1,i}B_{1,i} \\
& + 6zh^2(\phi_1+\psi_1)E_{1,ii} - 4\sqrt{z}hE_{1,ii}(B_1 - \sqrt{z}E_1')_{,jj} + 4\sqrt{z}hE_{1,ii}B_{1,jj} + 3zh^2 E_{1,ii}E_{1,jj} \\
& + 3zh^2 B_{1,i}B_{1,i} \Big] - \frac{\sqrt{-q}^{(2)}}{\kappa},
\end{aligned}
\tag{3.66}
$$

where $h \equiv b'/b$. Here some total derivatives have been omitted, and we have used the background equations of motion given in subsection 3.3.1.

On the other hand, action S_2 up to second order turns out to be:

$$
\begin{aligned}
S_2^{(2)} = \frac{1}{2}\int d^4x \frac{a^2b^2}{\kappa\sqrt{z}} \Big[& \frac{3}{2}(\phi_1^2 + \psi_1^2)(z-1) + \phi_1((z-1)(3\psi_1 - E_{1,ii}) - 6\psi_2 + 2E_{2,ii} - 2z\phi_2) \\
& + \psi_1(6\psi_2 - (z-1)E_{1,ii} - 2E_{2,ii} - 6z\phi_2) - \frac{1}{2}(z-1)(E_{1,ii}E_{1,jj} + B_{1,i}B_{1,i}) \\
& - 2E_{1,ii}(\psi_2 - z\phi_2 + E_{2,ii}) + 2\sqrt{z}B_{1,i}B_{2,i} \Big] \\
& - 2a^4\left(-\frac{1}{2}\phi_2^2 + \frac{1}{2}B_{2,i}B_{2,i} - 3\phi_2\psi_2 + \phi_2 E_{2,ii} + \frac{3}{2}\psi_2^2 - \frac{1}{2}E_{2,ii}E_{2,jj} - E_{2,ii}\psi_2\right).
\end{aligned}
\tag{3.67}
$$

Then, the total second-order scalar gravitational action is (3.66)+(3.67).

Gravitational tensor action

We must now replace (3.48) and (3.49) in $S_1 + S_2$. Notice that even though tensor metric perturbations do not couple to matter perturbations, this action will not be the same as the one of General Relativity because there is a different background matter.

As we explained in the previous chapter, an expansion in two polarisations $p = (+, \times)$ for the tensor modes is usually done (see (2.96)). For simplicity, we will choose a specific direction $\vec{k} = k\hat{z}$ so tensor perturbations lie in the xy plane. As a result, tensor metric perturbations can be written as:

$$ds_q^2 = b^2 \left[z^{-1}d\eta^2 - [(1 + h_{1+})dx^2 + (1 - h_{1+})dy^2 + dz^2 + 2h_{1\times}dxdy]\right] \tag{3.68}$$

$$ds_g^2 = a^2 \left[d\eta^2 - [(1 + h_{2+})dx^2 + (1 - h_{2+})dy^2 + dz^2 + 2h_{2\times}dxdy]\right] \tag{3.69}$$

where these perturbations depend on η and z.

We expand the gravitational action up to second order. For example, the determinants are:

$$\begin{aligned}
\sqrt{-q}^{(0)} &= \frac{b^4}{\sqrt{z}}, \\
\sqrt{-q}^{(1)} &= 0, \\
\sqrt{-q}^{(2)} &= -\frac{b^4}{2\sqrt{z}}\left[(h_{1+})^2 + (h_{1\times})^2\right],
\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}
\sqrt{-g}^{(0)} &= a^4, \\
\sqrt{-g}^{(1)} &= 0, \\
\sqrt{-g}^{(2)} &= -\frac{a^4}{2}\left[(h_{2+})^2 + (h_{2\times})^2\right],
\end{aligned} \tag{3.71}$$

Replacing these expressions into S_1 and S_2 we can write the tensor gravitational action $S_\mathrm{T} = S_1 + S_2$ up to second order as:

$$S_\mathrm{T}^{(2)} = S_\times^{(2)} + S_+^{(2)}, \tag{3.72}$$

where

$$S_p^{(2)} = \frac{1}{2}\int d^4x \frac{b^2}{2\sqrt{z}}\left[zh_{1p}'^2 + \frac{2}{\kappa}a^2h_{1p}^2 - h_{1p,z}^2 + \frac{2}{\kappa}a^2\left(h_{2p}^2 - 2h_{1p}h_{2p}\right)\right]. \tag{3.73}$$

We can see that the action $S_{\mathrm{T}}^{(2)}$ has two copies of the same action $S_p^{(2)}$ for each polarisation. Since there is no matter contribution to the tensor action, $S_{\mathrm{T}}^{(2)}$ corresponds to the total tensor second-order action.

Hydrodynamical scalar action

Now, we will calculate the second-order scalar matter action. The action for hydrodynamical matter, formed by particles with rest mass m_0, is:

$$S_{\mathrm{m}} = -\int d^4x\rho\sqrt{-g}, \tag{3.74}$$

where ρ is the total energy density, which can be written as

$$\rho = n[m_0 + \Pi], \tag{3.75}$$

where n is the number density of particles in the fluid, and Π is an energy associated to the fluid. For more information about this matter action, see Appendix B.

The matter action to second order in the longitudinal part χ of the shift vector χ^μ is:

$$\begin{aligned} S_{\mathrm{m}}^{(2)} = \int d^4xa^4 &\left[\frac{1}{2}\rho_0\phi_2^2 + p_0\left(\frac{3}{2}\psi_2^2 - 3\phi_2\psi_2 + \phi_2E_{2,ii} + \frac{1}{2}B_{2,i}B_{2,i} - \frac{1}{2}E_{2,ii}E_{2,jj} - E_{2,ii}\psi_2\right)\right.\\ &\left. + (\rho_0 + p_0)\left(\frac{1}{2}\chi_{,i}'^2 + B_{2,i}\chi_{,i}' + \phi_2\chi_{,ii}\right) - \frac{1}{6}(\rho_0 + p_0)(3\psi_2 - E_{2,ii} - \chi_{,ii}')^2\right]. \end{aligned} \tag{3.76}$$

Here we can see that even if we set $\chi = 0$, this second-order matter action would contribute to the total second-order action. This is because scalar perturbations are coupled to matter perturbations.

Finally, the total scalar second-order action $S_{\mathrm{s}}^{(2)}$ is (3.66)+(3.67)+(3.76). From this scalar action we can see that all perturbation fields, except for E_1, ψ_1, and χ, have no time derivatives. Fields with no time derivatives correspond to auxiliary variables, which can be worked out algebraically from their own equations of motion in terms of the fields that have time derivatives. Then, in principle, we could reduce the total action to an action only depending on E_1, ψ_1, and χ. Furthermore, if we use our gauge choice (3.57), the action would only depend on E_1 or, equivalently, on ζ. This is exactly what we will do in the next subsection.

Analogously, the total tensor second-order action $S_{\rm T}^{(2)}$ is (3.72). We can also find the auxiliary variables h_{2p}. Then the action could be reduced to one with only two degrees of freedoms: h_{1p}. We recall that since these fields are gauge-invariant they correspond to physical variables.

3.5.2 Reduced second-order action

Now, we will find a reduced second-order action containing only physical degrees of freedom: one scalar and two tensor. These reduced actions are to be quantised later.

Action for scalar perturbations

We have said that in this theory there is only one physical degree of freedom and we want to find an action for it. In order to do that, we will fix the gauge and eliminate all the auxiliary variables present in the total second-order action (3.66)+(3.67)+(3.76). In this way, we reduce the problem with 9 fields to a problem with only 1. To start the reduction process of the total second-order action we will primarily need all the equations of motion. To simplify calculations, we go to Fourier space and use our gauge choice (3.57) immediately. We perform the following replacement in the total action:

$$
\begin{aligned}
\phi_2(\eta,\vec{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}}\phi_2(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}}, \quad \phi_1(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\phi_1(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}},\\
E_2(\eta,\vec{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}}E_2(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}}, \quad E_1(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}E_1(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}},\\
B_2(\eta,\vec{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}}B_2(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}}, \quad B_1(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}B_1(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}},\\
\psi_2(\eta,\vec{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}}\psi_2(\eta,\vec{k})e^{i\vec{k}\cdot\vec{x}},
\end{aligned}
$$

and define $k^2 \equiv \vec{k}\cdot\vec{k} = k_x^2 + k_y^2 + k_z^2$. Here, we didn't show χ nor ψ_1, because they were set to zero in our gauge choice. For simplicity, we will omit the dependence of these fields.

Varying the whole action $S_{\rm s}^{(2)}$ with respect to these scalar perturbation fields in Fourier space, we obtain the following equations of motion:

$$
\begin{aligned}
\delta\phi_2: &\quad \left(\phi_1 - \phi_2 + 3 + E_1 k^2\right) z - E_2 k^2 - 3\psi_2 = 0,\\
\delta\psi_2: &\quad 3(\phi_2 - \phi_1) - \left(E_2 k^2 + 3\psi_2\right) z + E_1 k^2 = 0,\\
\delta E_2: &\quad 3(\phi_1 - \phi_2) + k^2 \left(3E_1 - 4E_2\right) + z\left(3\psi_2 + k^2 E_2\right) = 0,\\
\delta B_2: &\quad B_2 - \sqrt{z} B_1 = 0,\\
\delta\phi_1: &\quad 2z\kappa h E_1' k^2 - (3a^2 - 2b^2)\phi_1 + a^2 z\phi_2 + 3a^2\psi_2 - 2(\kappa h\sqrt{z} B_1 + \frac{1}{2}a^2(E_1 - E_2))k^2 = 0,\\
\delta E_1: &\quad 2\kappa z h\phi_1' - (-2b^2 + a^2)\phi_1 + a^2(\psi_2 - k^2 E_2 + k^2 E_1 - z\phi_2) = 0,\\
\delta B_1: &\quad a^2\sqrt{z} B_2 - B_1 a^2 + 2\kappa\sqrt{z} h\phi_1 = 0.
\end{aligned}
$$

From the first 4 equations, we can obtain all the perturbation fields for g: E_2, B_2, ϕ_2, and ψ_2, in terms of the perturbation fields of q:

$$
\begin{aligned}
\phi_2 &= \frac{(3 + z^2)\phi_1 + k^2 E_1 (z+1)(z-1)}{3 + z^2},\\
\psi_2 &= \frac{-k^2 (z-1)(1/3z - 1)E_1}{3 + z^2},\\
B_2 &= \sqrt{z} B_1,\\
E_2 &= E_1.
\end{aligned}
$$

We replace these results in the 3 equations left. ϕ_1 and B_1 can also be worked out in terms of E_1:

$$
\begin{aligned}
\phi_1 &= -\frac{(z-1)(\kappa z h k^2 E_1' + \frac{1}{2}k^2 E_1 (z-1)a^2)a^2}{(3(z-1)a^2 + 2\kappa k^2)\kappa h^2 z},\\
B_1 &= 2\frac{\kappa h z E_1' k^2 + \frac{1}{2}k^2 a^2 E_1 (z-1)}{(3a^2(z-1) + 2\kappa k^2)h\sqrt{z}}.
\end{aligned}
$$

Finally, we have obtained all fields in terms of E_1. If we write these fields in terms of ζ, by using relation (3.61), and replace them in the total second-order action obtained in the previous subsection, we will get a final reduced action in Fourier space for the field ζ:

$$
S_{\rm s}^{(2)} = \frac{1}{2}\int c_1 \left(\zeta'^2 - c_2\zeta^2\right) d^3k d\eta, \tag{3.77}
$$

where the coefficients c_1 and c_2 depend on time η and k:

$$c_1 = \frac{18b^2a^2(z-1)\sqrt{z}}{X}, \tag{3.78}$$

$$c_2 = 9b^2a^2 \Big\{ 2\mathcal{H}(z^2+3)\kappa \left[9(z-2)(z-1)^2a^4 + 12k^2\kappa(z-1)(z-2)a^2 + 2k^4\kappa^2(z-3)\right] + z'(z^2+3)\left[9(z-1)^2a^4 + 12k^2\kappa(z-1)a^2 + 2\kappa^2k^4\right] + hX\kappa\left[12z(z-1)(1+z^2)a^2 + (3+5z+z^2+7z^3)k^2\kappa\right]\Big\} / \left\{X^2\kappa^2h\sqrt{z}(z^2+3)c_1\right\}, \tag{3.79}$$

where $X \equiv 3(z-1)a^2 + 2k^2\kappa$. Notice that since the background is homogeneous and isotropic, the field ζ depends on η and $||\vec{k}||$.

For later use, we will perform a redefinition of the field ζ:

$$\zeta(\eta, k) = \frac{v(\eta, k)}{\sqrt{c_3}}; \quad c_3 = \frac{c_1}{b^2\sqrt{z}}, \tag{3.80}$$

such that the scalar action looks like that of a scalar field coupled to the metric q with a time-dependent mass:

$$S_{\mathrm{s}}^{(2)} = \frac{1}{2}\int b^2\sqrt{z}\left(v'^2 - c_4v^2\right) d^3k d\eta. \tag{3.81}$$

Here, c_4 is a time-dependent factor related to c_1, c_2, and c_3:

$$c_4 = c_2 + \frac{1}{4}\left(\frac{c_3'}{c_3}\right)^2 - \frac{1}{2}\left(\frac{c_3'}{c_3}\right)' - \frac{1}{2}\left(\frac{c_1'}{c_1}\right)\left(\frac{c_3'}{c_3}\right). \tag{3.82}$$

Action for tensor perturbations

Analogously, we go to Fourier space

$$h_{2p}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} h_{2p}(\eta, \vec{k}) e^{i\vec{k}\cdot\vec{x}}; \quad h_{1p}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} h_{1p}(\eta, \vec{k}) e^{i\vec{k}\cdot\vec{x}},$$

and calculate the equations of motion from $S_{\rm T}^{(2)}$:

$$\delta h_{1p}: \quad h''_{1p} + \left(2h + \frac{z'}{2z}\right) h'_{1p} + \left(\frac{k^2}{z} + \frac{2}{\kappa z}a^2\right) h_{1p} - \frac{2}{\kappa z}a^2 h_{2p} = 0,$$

$$\delta h_{2p}: \quad h_{2p} - h_{1p} = 0.$$

Using the last equation we can write h_{2p} in terms of h_{1p} and replace them in the action calculated in the previous section, to obtain the reduced second-order action in Fourier space for tensor perturbations:

$$S_{\rm T}^{(2)} = S_{\times}^{(2)} + S_{+}^{(2)}, \tag{3.83}$$

where

$$S_p^{(2)} = \frac{1}{2} \int b^2 \sqrt{z} \left(h'^2_{1p} - \frac{k^2}{z} h^2_{1p}\right) d^3k d\eta. \tag{3.84}$$

Notice this action looks like that of a scalar field coupled to the metric q with a time-dependent mass. Since the two fields h_{1p} are the same, from now on, we will call them $h = h_{1+} = h_{1\times}$ and work with the action $S_p^{(2)}$ for only one of them.

Finally, we calculated an action for scalar and tensor perturbations containing only the physical degrees of freedom, one scalar and two tensor. Next, in order to understand the fields we are working with, we will study their classical behaviour.

3.5.3 Classical solution

Let us study the behaviour of the classical solution of $\zeta(\eta, k)$ and $h(\eta, k)$.

Scalar solution

First, Let us analyse the behaviour of ζ in the Eddington period, where a is near the minimum a_B. The equation of motion for ζ can be approximated in both cases of κ:

- Case $\kappa > 0$:

$$\zeta'' = 0 \quad \Rightarrow \quad \zeta(\eta, k) = A_k \eta + B_k, \tag{3.85}$$

where A_k and B_k are some integration constants, depending on k.

- Case $\kappa < 0$:

$$\zeta'' + \frac{2}{\eta}\zeta' + \frac{k^2}{3\beta^2\eta^2}\zeta = 0 \quad \Rightarrow \quad \zeta(\eta, k) = A_k\eta^{n_+} + B_k\eta^{n_-}, \tag{3.86}$$

where $\beta \equiv a_B\sqrt{2/(3|\kappa|)}$ and $n_\pm \equiv -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4k^2/(3\beta)}$. Also, here we have set $a(\eta = 0) = a_B$.

From these two solutions (3.85) and (3.86) we can see that in both cases there is a divergence as $a \to a_B$. In the case of $\kappa > 0$ we have a linear divergence in η, and for $\kappa < 0$ a polynomial one.

Figure 3.3 shows the evolution of ζ (blue and gold lines) as a function of the conformal time η, for both cases of κ during the radiation-dominated era. These plots represent a numerical solution with arbitrary initial conditions, so they show a typical behaviour of ζ. In the case of $\kappa < 0$, we set $a(0) = a_B$. In this figure we observe the divergences shown by equations (3.85) and (3.86) in the Eddington period, when $\eta \to -\infty$ for $\kappa > 0$ and $\eta = 0$ for $\kappa < 0$. During the Einstein period, both solutions are equal.

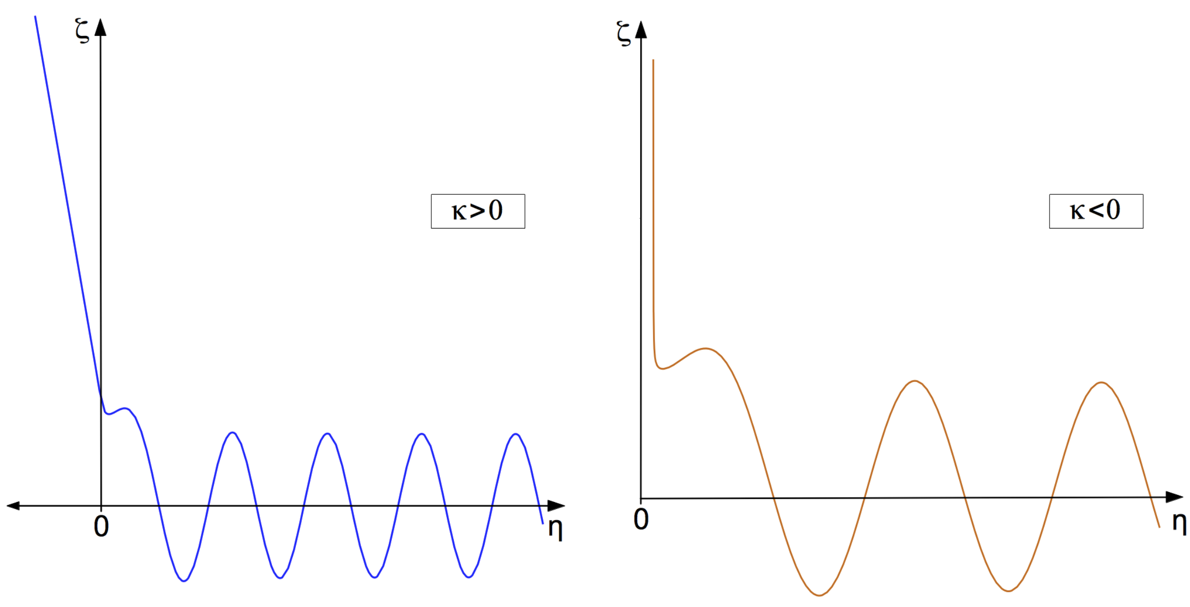

Figure 3.3: Evolution of ζ (blue and gold lines) as a function of η during the radiation-dominated era, with arbitrary initial conditions. A divergence as $a \to a_B$ is observed in both cases of κ.

Tensor solution

The equation of motion for h coming from action (3.84) is:

$$h'' + \frac{(b^2\sqrt{z})'}{b^2\sqrt{z}}h' + \frac{k^2}{z}h = 0. \tag{3.87}$$

The behaviour of h near a_B is:

- Case $\kappa > 0$:

$$h'' = 0 \quad \Rightarrow \quad h(\eta, k) = A_k\eta + B_k. \tag{3.88}$$

- Case $\kappa < 0$:

$$h'' + \frac{2}{\eta}h' + \frac{k^2}{3\beta^2\eta^2}h = 0 \quad \Rightarrow \quad h(\eta, k) = A_k\eta^{n_+} + B_k\eta^{n_-}. \tag{3.89}$$

From (3.88) and (3.89) we can see that h has the same divergence as ζ.

Figure 3.4 shows the evolution of h (blue and gold lines) as a function of the conformal time η, for both cases of κ. These plots represent a numerical solution with arbitrary initial conditions, so they show a typical behaviour of h. Again, we see the divergences in the Eddington period, and the same behaviour of the fields for late radiation.

Finally, we see there is a scalar and tensor instability in the limit $a \to a_B$ in both cases of κ. For $\kappa > 0$, the divergence is linear in the infinite past. For $\kappa < 0$, the divergence is a power of time when the bounce occurs. This characteristic had already been observed for the tensor perturbations in Escamilla-Rivera, Bañados and Ferreira (2012).

Since the amplitude of these fields was large near a_B, at some point in time, the linear theory of perturbations breaks down because $\delta g_{\mu\nu}/g^{(0)}_{\mu\nu} \ll 1$ is violated. This problem can be caused by the linear perturbation theory (corrections of higher order could change this behaviour) or by the EBI theory itself. A similar problem appears in inflation, where the physical fields diverge in the Big Bang. However, we will consider only the region where the linear theory is still valid, and forget about this problem.

Having studied scalar and tensor perturbations classically, the next step is to quantise them in order to obtain a prediction for the power spectrum at the second horizon crossing, in analogy to the previous chapter.

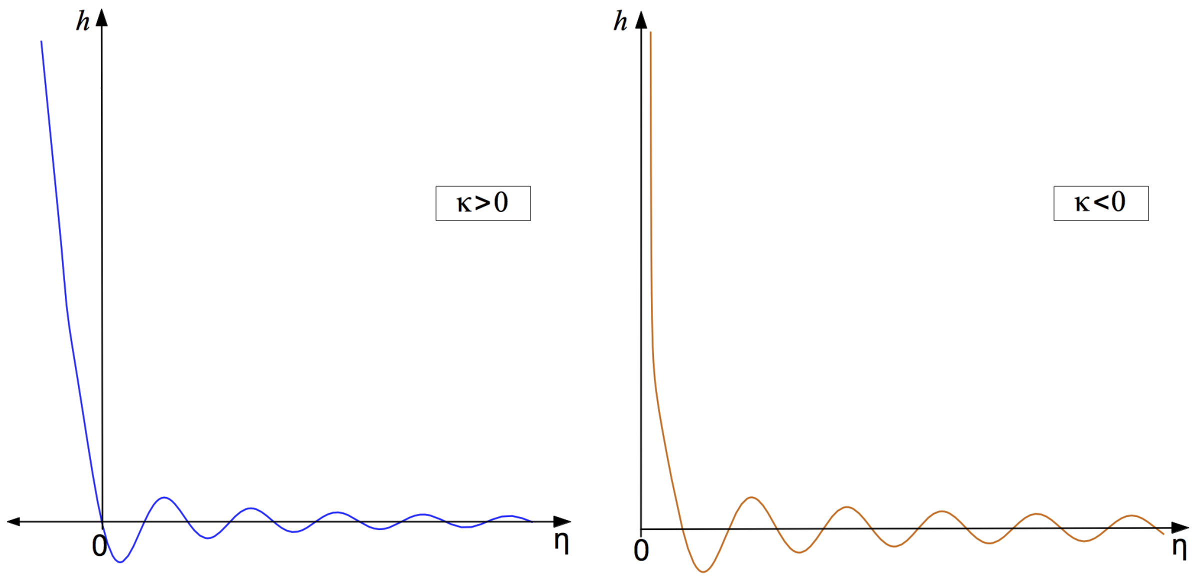

Figure 3.4: Evolution of h (blue and gold lines) as a function of η during the radiation-dominated era, with arbitrary initial conditions. A divergence as $a \to a_B$ is observed in both cases of κ.

3.5.4 Alternative inflation

In the previous chapter we showed that primordial perturbations were produced by quantum fluctuations of the inflationary field and its information was contained in its two-point correlation function. These perturbations were originated in their ground state at sub-Hubble scales, specifically in the Bunch-Davies vacuum. When they enter the horizon, the quantum states become characterised by a large occupation number and the system collapses into classical states. These classical states represent a random spectrum of perturbations whose variance is given by the quantum correlations evaluated at this quantum-to-classical transition point.

However, different vacua (or, equivalently, different initial conditions) for perturbations could have been given for inflation (Allen (1985)), which would have given different results for the power spectra at the second horizon crossing point. It seems that we cannot go forward on this problem without a precise knowledge of a theory of quantum gravity. This theory could tell us exactly how the Universe leaves the Planck scale, and then give a correct initial condition for the fluctuations. Since some cosmologically relevant wavelengths, at some early stage, could have been trans-Planckian (quantum regime beyond the reach of our theories) during inflation, there is no justification for any choice of vacuum, which casts doubts on the validity of the cosmological perturbation predictions of inflation (Armendariz-Picon 2007). In view of this difficulty, along with the fact that the inflationary field is still unknown, among others, the question is whether the inflationary explanation is unique.

One alternative proposal is given by Hollands and Wald (Hollands and Wald (2008)). They point out that inflation does not really solve the flatness problem, since one still needs special initial conditions to initiate the inflation period. This is the motivation to develop a mechanism that results in a scale-free power spectrum for fluctuations in General Relativity, without assuming the existence of a fundamental scalar field as matter. Let us describe the main ideas behind this alternative proposal.

Hollands-Wald mechanism

Let us assume that there is a fundamental length called l_0. Semiclassical physics applies to phenomena on spatial scales larger that l_0, so modes emerge from an unknown fundamental description of spacetime at that scale. Therefore, we may assume that a perturbation with physical wavelength λ_{ph} is effectively born at l_0 in the ground state of a flat spacetime. Since λ_{ph} grows in time, the perturbations are continuously being created.

Hollands and Wald applied the previous mechanism for scalar perturbations in General Relativity during the early Universe, which they consider as the radiation-dominated era, and obtained a scale-free power spectrum at the second horizon crossing. The initial condition set with this method for perturbations is different than the Bunch-Davies vacuum. In order to have the correct amplitude for the power spectrum, l_0 is chosen to be $l_0 = 10^5 l_p$, with l_p the Planck scale.

3.5.5 Initial conditions

To quantise the perturbations, we promote them to quantum operators, in the same way we did in the previous chapter:

$$\hat{\zeta}(\eta, \vec{k}) = \zeta_{\vec{k}}(\eta) a_{\vec{k}} + \zeta^*_{\vec{k}}(\eta) a^\dagger_{\vec{k}}, \quad \hat{h}(\eta, \vec{k}) = h_{\vec{k}}(\eta) a_{\vec{k}} + h^*_{\vec{k}}(\eta) a^\dagger_{\vec{k}}, \tag{3.90}$$

where $a_{\vec{k}}$ with $a^{\dagger}_{\vec{k}}$ are the annihilation and creation operators, respectively. Functions $\zeta_{\vec{k}}(\eta)$ and $h_{\vec{k}}(\eta)$ are some solution to the classical equations of motion for ζ and h, whose normalisations will be given by initial conditions.

If we followed the same ideas as in the previous chapter to quantise the perturbations, we would set the initial conditions by minimising the vacuum energy in the limit $a \to a_B$, as analogy to the Bunch-Davies vacuum, which minimises the vacuum energy in the asymptotic past $\eta \to \infty$. However, with the arguments given in the previous subsection, we know that it would be valid also to fix another initial condition. In fact, we will follow the Hollands-Wald mechanism to put initial conditions in the quantum solutions $\hat{\zeta}(\eta, \vec{k})$ and $\hat{h}(\eta, \vec{k})$, albeit with a slight modification. We will define[1] $l_0 = \sqrt{|\kappa|} = 10^4 l_p$, and assume that a mode with comoving wavenumber k is created at η_* such that $b_*/k = l_0$. Notice that we have defined this relation[2] with the scale factor of the metric q, instead of g. Then, the initial condition for the perturbations will be that they are in the ground state at η_*.

We can estimate the scale at which η_* occurs if $l_0 \approx 10^{-31} m$ and $k \approx 10^{-26} m^{-1}$ is of order of our present cosmological horizon:

- Scale factor: $b \approx 10^{-57}$ and $a \approx a_B(1 + 10^{-110})$
- Energy density: $\rho_0 \approx 10^{-8} \rho_p$, where ρ_p is the Planck energy density.

We can see here that η_* occurs in the Eddington period, when λ is a sub-Hubble scale. This is valid for all cosmologically relevant scales.

Next, we will calculate the quantum perturbations $\hat{\zeta}$ and $\hat{h}$ for the case of $\kappa > 0$ at η_*, using the Hollands-Wald mechanism[3]. In order to do that, we will apply the standard QFT rules for a scalar field in an adiabatic approximation to the actions calculated previously

---

[1]We have chosen this order of magnitude of l_0 in order to have the correct order of magnitude for the power spectrum of scalar perturbations.

[2]This modification is motivated by the form of the second-order action for tensor perturbations. In General Relativity the action for h has the form of an action for a scalar field coupled to the metric g, but in the EBI theory it appears coupled to q.

[3]We will not consider the case of $\kappa < 0$ because it is not well behaved near η_*. In particular, it is found that the mass-like term in the action is positive, instead of negative as in an action for a massive scalar field.

for v and h at η_*, and use relation (3.80) to find the quantum solution for ζ. This result will give the initial value of the quantum perturbations, which, as we will discuss later, must be extrapolated in time upto the Einstein period of radiation, where the power spectra will be calculated.

Scalar case

Since the transition to a semiclassical theory happens in the Eddington period for all relevant scales, we approximate $S_{\rm s}^{(2)}$ when $a \approx a_B$ for $\kappa > 0$:

$$S_{\rm s}^{(2)} \approx \frac{1}{2} \int d\eta d^3k \; 4a_B^2 \left(v'^2 - \frac{5k^2}{9a_B}(a - a_B) v^2 \right). \tag{3.91}$$

We write the quantum solution of v as:

$$\hat{v}(\eta, \vec{k}) = v_{\vec{k}} a_{\vec{k}} + v_{\vec{k}}^* a_{\vec{k}}^\dagger, \tag{3.92}$$

By making an adiabatic approximation[4] near η_* in action (3.91), we can use the standard QFT rules to quantise v and then write $v_{\vec{k}}$ at η_* as:

$$v_{\vec{k}}(\eta_*) = \frac{1}{\sqrt{8a_B^2 \omega_*}} e^{i\omega_* \eta_*}; \quad \omega_* = \sqrt{\frac{5k^2}{9a_B}(a_* - a_B)}. \tag{3.93}$$

However, in η_* the following holds:

$$\frac{b_*}{k} \approx \frac{2a_B^{3/4}(a_* - a_B)^{1/4}}{k} = \sqrt{\kappa}. \tag{3.94}$$

Then, ω_* and η_* are explicitly,

$$\omega_* = \frac{\sqrt{5}k^3\kappa}{12a_B^2}; \quad \eta_* = \frac{2\sqrt{3\kappa}}{\sqrt{2}a_B} \ln\left(\frac{k\sqrt{\kappa}}{2a_B} \right). \tag{3.95}$$

The relation between v and ζ is given by (3.80), which in the Eddington period is

[4] The adiabatic approximation consists in taking an interval of time small enough to allow us to consider the background functions as effectively constant. In our specific calculations, we take an interval of time around η_*.

approximately:

$$\zeta \approx -\sqrt{\frac{1}{6}}v. \tag{3.96}$$

Therefore, the solution $\zeta_{\vec{k}}$ at η_* is:

$$\zeta_{\vec{k}}(\eta_*) = -\sqrt{\frac{1}{4\sqrt{5}\kappa k^3}} e^{i\omega_*\eta_*}. \tag{3.97}$$

Tensor case

Analogously, we approximate the tensor action for h in the Eddington period for $\kappa > 0$:

$$S_{\mathrm{T}}^{(2)} \approx \frac{1}{2}\int d\eta d^3k\; 4a_B^2 \left(h'^2 - \frac{k^2}{a_B}(a - a_B)h^2\right), \tag{3.98}$$

and write $h_{\vec{k}}$ at η_* as:

$$h_{\vec{k}}(\eta_*) = \frac{1}{\sqrt{8a_B^2\tilde{\omega}_*}} e^{i\tilde{\omega}_*\eta_*}; \quad \tilde{\omega}_* = \sqrt{\frac{k^2(a_* - a_B)}{a_B}}. \tag{3.99}$$

Using (3.94) we find that $\tilde{\omega}_*$ is explicitly:

$$\tilde{\omega}_* = \frac{\kappa k^3}{4a_B^2}, \tag{3.100}$$

and then

$$h_{\vec{k}}(\eta_*) = \sqrt{\frac{1}{2\kappa k^3}} e^{i\tilde{\omega}_*\eta_*}. \tag{3.101}$$

Finally, we have obtained initial conditions (3.97) and (3.101) at η_* for scalar and tensor quantum perturbations, respectively. Notice that in this subsection we made the quantisation of the scalar and tensor actions written in the form of a scalar field coupled to the metric q. In general, we could have written the actions in a different form, say as a scalar field coupled to the metric g, which would have given a different result for the initial conditions. This fact is related to the problem of defining different vacua, which we mentioned previously and need further analysis. However, in this theory, when making the quantisation as we just did we obtain the correct form for the power spectra, as we will show later.

3.5.6 Spectrum of perturbations

In the previous chapter we mentioned that an initial nearly scale-invariant power spectrum for the gauge-invariant gravitational potential Φ at the second horizon crossing, as it is in inflation, is in agreement with the observations for CMB temperature anisotropies and matter distribution today, when the extrapolation in time is made classically in General Relativity. We also found a nearly scale-invariant power spectrum for tensor perturbations. For that reason, our objective is to calculate the power spectrum of Φ and h at the second horizon crossing in the EBI theory.

Scalar spectrum

The gauge-invariant gravitational potential Φ is defined by

$$\Phi \equiv \phi_2 + a^{-1}[a(B_2 - E_2)]'. \tag{3.102}$$

We will calculate its power spectrum in the same way as in the previous chapter: we will first find the power spectrum of the curvature perturbation ζ at the second horizon crossing and then relate this value with the power spectrum of Φ during radiation/matter-dominated eras, by using (2.95). This relation is valid now because during the Einstein period for super-Hubble scales $\zeta \simeq \mathcal{R}$ (see Riotto (2002)).

Since, as we said in the previous chapter, $\mathcal{R}$ freezes for all super-Hubble scales in General Relativity, then ζ freezes for all super-Hubble scales in the Einstein period. Consequently, the power spectrum of ζ, for a super-Hubble scale, at any time of the Einstein period of radiation, is the same than at the second horizon crossing.

Therefore, in order to find the power spectrum of ζ at the second horizon crossing time, we will extrapolate $\zeta_{\vec{k}}$ numerically in time using the classical equation of motion and the initial condition (3.97), but also its time derivative at η_*. After that, we will evaluate the numerical solution in a particular time during the Einstein period of radiation, say $c\eta = 100\,\mathrm{m}$, and calculate $|\zeta_{\vec{k}}|^2$ there. Since the power spectrum of $\zeta_{\vec{k}}$ is

$$\mathcal{P}_\zeta(\eta, k) = \frac{k^3}{2\pi^2}|\zeta_{\vec{k}}(\eta)|^2, \tag{3.103}$$

the described calculations will give us the power spectrum at $c\eta = 100\,\mathrm{m}$, i.e. at the second horizon crossing time. We are interested in knowing the dependence of $\mathcal{P}_\zeta$ in k for super-Hubble scales, so we apply this numerical process to different orders of magnitude $k = 10^i\,\mathrm{m}^{-1}$, where i goes from -26 to -16. Figures (3.5) and (3.6) show this result. Figure (3.5) shows the logarithmic value in base 10 of $|\zeta|^2k^3$, which is the power spectrum, at $c\eta = 100\,\mathrm{m}$ for the 11 different values of k. We observe that the power spectrum is scale-invariant because there is no dependence on the value of k. Only to check this characteristic, we make the two plots in figure (3.6). Here we have the logarithmic value of $|\zeta|^2k^2$ (asterisks), where a clear dependence on k on the form of $1/k$ is observed. We also have the logarithmic value of $|\zeta|^2k^4$ (crosses), where a linear dependence on k is observed.

Finally, we conclude that the power spectrum of ζ at the second horizon crossing has the following form:

$$\mathcal{P}_\zeta(k) = A_\zeta^2 k^{n_\mathrm{s}-1}, \tag{3.104}$$

where A_ζ is an amplitude and n_s is the scalar spectral index. Because of relation (2.95), this last result means that the power spectrum $\mathcal{P}_\Phi$ is also scale-invariant. From the numerical calculations we find that for the 11 wavenumbers:

$$\mathcal{P}_\zeta \sim 10^{-9}; \quad n_\mathrm{s} - 1 \sim 10^{-28} - 10^{-46}, \tag{3.105}$$

where the range for n_s appears because it depends on k. From here we can see that the dependence of the power spectrum on k is incredibly small, so it can be neglected.

Tensor spectrum

To calculate the power spectrum of h at the second horizon crossing we do the same as for the scalar field ζ. We perform numerical calculations to find $|h_{\vec{k}}|^2$ at $c\eta = 100\,\mathrm{m}$, i.e. for super-Hubble scales. Figures (3.7) and (3.8) show the results. Again, from these three plots we conclude that the power spectrum is scale-invariant:

$$\mathcal{P}_\mathrm{T}(k) = \frac{2k^3}{\pi^2}|h_{\vec{k}}|^2 = A_\mathrm{T}^2 k^{n_\mathrm{T}}, \tag{3.106}$$

where A_T is an amplitude and n_T is the tensor spectral index. From the calculations we

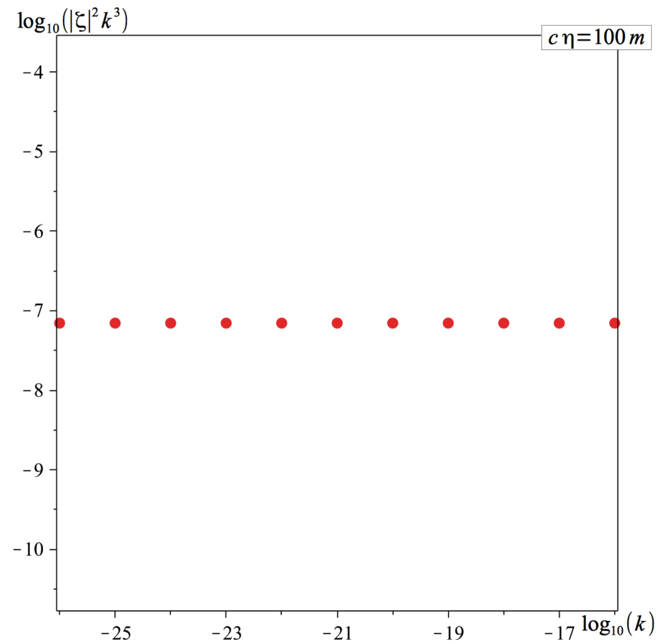

Figure 3.5: Logarithmic value of $|\zeta|^2 k^3$ as a function of $\log_{10}(k)$, with k being the comoving wavenumber in MKS units. A scale-invariant function is observed.

find that for the 11 scales,

$$\mathcal{P}_{\rm T} \sim 10^{-8}; \quad n_{\rm T} \sim 10^{-28} - 10^{-46}, \tag{3.107}$$

where, again, the range for $n_{\rm T}$ appears because it depends on k. From here we can see that the dependence of the power spectrum on k is incredibly small, so it can be neglected.

Now, we can compare our estimations for the power spectra of scalar and tensor perturbations with observations, which are in agreement with a scalar power spectrum of the form (see Komatsu et al. (2011)):

$$\mathcal{P}_{\mathcal{R}}(k) = A_{\mathcal{R}}^2 \left(\frac{k}{k_0}\right)^{n_{\rm s}-1}, \tag{3.108}$$

where $k_0 = 0.002$ Mpc^{-1}, and

$$\mathcal{P}_{\mathcal{R}}(k_0) = (2.430 \pm 0.091) \times 10^{-9}; \quad n_{\rm s} - 1 = -0.032 \pm 0.012. \tag{3.109}$$

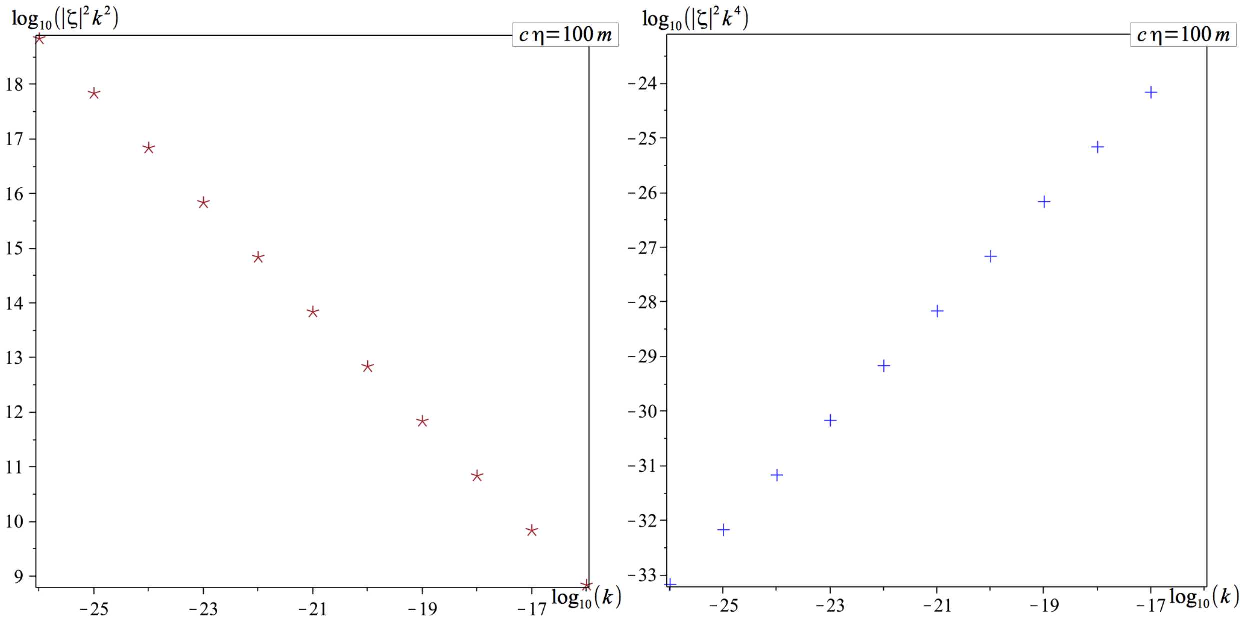

Figure 3.6: Logarithmic value of $|\zeta|^2 k^2$ (asterisks) and $|\zeta|^2 k^4$ (crosses) as a function of $\log_{10}(k)$, with k being the comoving wavenumber in MKS units.

From (3.105) we can see that the amplitude of the scalar power spectrum has the right order of magnitude. Since the experimental data agree with a constant scalar spectral index, we expected to have, at least, a small dependence of n_s on k but from the numerical calculation we can see that it roughly goes like k^2, and from (3.105) we see that it has the wrong order of magnitude.

Also, a maximum value for the tensor-to-scalar ratio $r(k)$ has been measured for k_0 (see Komatsu et al. (2011)):

$$r(k_0) \equiv \frac{\mathcal{P}_\mathrm{T}(k_0)}{\mathcal{P}_\mathcal{R}(k_0)} < 0.24, \tag{3.110}$$

which in our case is near to 10. The disagreement in the scalar spectral index n_s and the ratio r could be a problem of the theory or the vacuum choice. In either case, further research is required.

To sum up, in this section we have calculated a second-order action for all the perturbation fields. After that, we eliminated all the non-physical fields in order to find actions containing only the physical scalar and tensor perturbations. We found only 1 physical

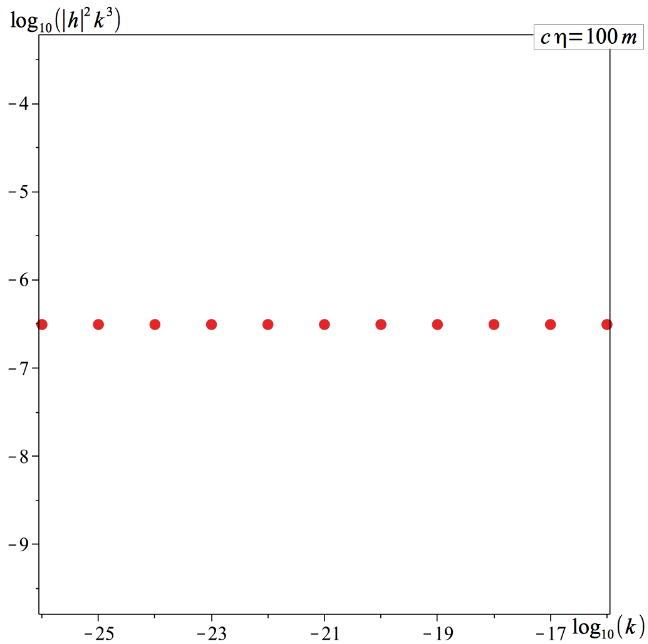

Figure 3.7: Logarithmic value of $|h|^2k^3$ as a function of $\log_{10}(k)$, with k being the comoving wavenumber in MKS units. A scale-invariant function is observed.

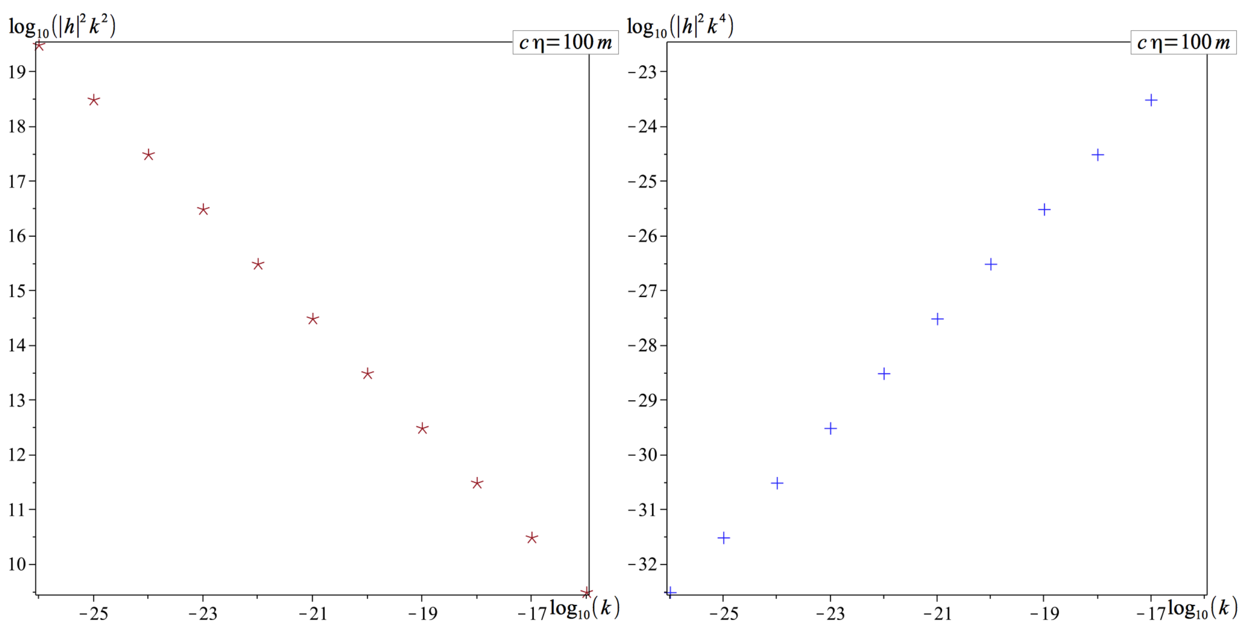

Figure 3.8: Logarithmic value of $|h|^2k^2$ (asterisks) and $|h|^2k^4$ (crosses) as a function of $\log_{10}(k)$, with k being the comoving wavenumber in MKS units.

scalar degree of freedom ζ and two equal tensor degrees of freedom h. We showed the classical behaviour of these fields during the radiation-dominated era, and observed an instability as $a \to a_B$ for both types of perturbations, and in both cases of κ. We quantised canonically the physical fields and used the Hollands-Wald mechanism to fix initial conditions on the quantum solutions. Finally, we extrapolated the quantum solutions in time for 11 different wavenumbers in order to calculate the power spectra at the second horizon crossing time. We found a scale-invariant power spectrum for both scalar and tensor perturbations, although not all the numerical estimations are in agreement with observations. Notice that in our calculations perturbations are also Gaussians, which results from considering only first order perturbations, and also assumed to be adiabatic.

Chapter 4

Conclusions

General Relativity is today the most successful theory of gravity. However, in the context of cosmology, it is not exempt of problems. It predicts the event of the Big Bang, which is a physical divergence since physical quantities such as the energy density diverge. In addition, the theory of inflation in General Relativity, which solves the horizon problem and the flatness problem, and gives an explanation for structure formation, has been criticised mainly for its assumption of a fundamental unknown scalar field with some peculiar characteristics.

In an attempt to eliminate the Big Bang divergence, the Eddington-Born-Infeld (EBI) theory appears. It is a classical gravitational theory which introduces modifications to General Relativity in regions with high curvature. This characteristic brings important consequences during the early Universe. For an expanding, flat, homogeneous and isotropic Universe coupled to a perfect fluid, the theory predicts two possible evolutions of the Universe, both of them with no Big Bang, i.e. the Universe has an infinite past, with a minimum value a_B for the scale factor and an early period of accelerated expansion. This result is within the assumption that the early Universe was dominated by radiation, as it is usually done. As a consequence, at least one of the possible behaviours of the Universe solves the horizon and flatness problem.

Since the horizon and flatness problem were the main reason to develop the inflation theory, the previously mentioned cosmological results in the EBI theory motivated us to consider the EBI theory as a good cosmological model and, particularly, a possible alternative to the inflation theory. For this reason, we studied inhomogeneities in our Universe by assuming that they were generated by primordial quantum first order perturbations in the classical cosmological background recently described. We used the standard classification for perturbations: scalar, vector and tensor. We found that vector perturbations seem to be not relevant because they decay in an expanding Universe. Consequently, we focused on scalar and tensor perturbations only. By making a study of the gauge symmetry present in the EBI theory, we were able to write an action containing the physical perturbation degrees only: one scalar and two equal tensor perturbations. We made a classical study of these fields and found a divergence as $a \to a_B$. This divergence, which is also present in the inflation theory, is not necessarily bad because it could be a result of the perturbation theory.

Finally, we used the canonical formalism to quantise these scalar and tensor physical fields. We argued that there were some ambiguities in making a vacuum choice and, consequently, we were free to use any procedure we wanted. We decided to use the Hollands-Wald mechanism. With this procedure, we found a scale-free power spectra for scalar and tensor perturbations, which is expected from observations. However, our numerical estimations do not fit all the experimental values. This last disagreement can be caused by a problem of the EBI theory or the quantisation procedure, and it will be left as future work.

Appendix A

Eddington-Born-Infeld actions

In this appendix, we will show that actions (3.10) and (3.13) are equivalent. Let's consider the case with $\lambda = 1$ (no cosmological constant). Consider

$$S[q, g, \Gamma, \chi] = -\frac{1}{2}\int d^4x\sqrt{-q}\left(q^{\mu\nu}R_{\mu\nu}(\Gamma) + \frac{2}{\kappa}\right) - \frac{1}{\kappa}\left(\sqrt{-q}q^{\mu\nu}g_{\mu\nu} - 2\sqrt{-g}\right) + S_{\rm m}[\chi, g]. \tag{A.1}$$

The equation of motion coming from variations of q is:

$$\delta q: \quad -\frac{1}{2}q_{\mu\nu}\left(q^{\alpha\beta}R_{\alpha\beta}(\Gamma) - \frac{2}{\kappa}\right) + R_{\mu\nu}(\Gamma) + \frac{1}{\kappa}g_{\mu\nu} - \frac{1}{2\kappa}q_{\mu\nu}(q^{\alpha\beta}g_{\alpha\beta}) = 0. \tag{A.2}$$

We now rewrite this equation. We take its trace:

$$q^{\mu\nu}R_{\mu\nu}(\Gamma) = -\frac{4}{\kappa} + \frac{1}{\kappa}q^{\mu\nu}g_{\mu\nu} \tag{A.3}$$

and replace it into (A.2), obtaining

$$q_{\mu\nu} = g_{\mu\nu} - \kappa R_{\mu\nu}(\Gamma). \tag{A.4}$$

Here, we have obtained an algebraic equation for $q_{\mu\nu}$, thus we can replace q, as a function of g and Γ, into the action (A.1) and obtain an action depending only on g, Γ and χ:

$$S[g, \Gamma, \chi] = \frac{1}{\kappa}\int d^4x\left(\sqrt{|g_{\mu\nu} - \kappa R_{\mu\nu}(\Gamma)|} - \sqrt{-g}\right) + S_{\rm m}[\chi, g], \tag{A.5}$$

which is exactly the same as (3.10).

Now, we will show that (A.1) is also equivalent to (3.13), and consequently the latter is equivalent to (3.10). To do this, we must get Γ as a function of the other variables, by means of its equation of motion, and replace it back into the action. We vary action (A.1) with respect to Γ:

$$\begin{aligned} \delta S &= \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= \int d^4x \sqrt{-q} q^{\mu\nu} \left(\nabla_\alpha \delta \Gamma^\alpha{}_{\mu\nu} - \nabla_\nu \delta \Gamma^\alpha{}_{\mu\alpha} \right) \\ &= \int d^4x \left[\nabla_\alpha (\sqrt{-q} q^{\mu\nu} \delta \Gamma^\alpha{}_{\mu\nu}) - \nabla_\nu (\sqrt{-q} q^{\mu\nu} \delta \Gamma^\alpha{}_{\mu\alpha}) \right] \\ &- \int d^4x \left[\nabla_\alpha (\sqrt{-q} q^{\mu\nu}) - \nabla_\beta (\sqrt{-q} q^{\mu\beta} \delta^\nu{}_\alpha) \right] \delta \Gamma^\alpha{}_{\mu\nu} \\ &= \int d^4x \left[\partial_\alpha (\sqrt{-q} q^{\mu\nu} \delta \Gamma^\alpha{}_{\mu\nu}) - \partial_\nu (\sqrt{-q} q^{\mu\nu} \delta \Gamma^\alpha{}_{\mu\alpha}) \right] \\ &- \int d^4x \left[\nabla_\alpha (\sqrt{-q} q^{\mu\nu}) - \nabla_\beta (\sqrt{-q} q^{\mu\beta}) \delta^\nu{}_\alpha \right] \delta \Gamma^\alpha{}_{\mu\nu}. \end{aligned} \tag{A.6}$$

The first integral is a boundary term so we will disregard it. Then, the equation of motion is:

$$\nabla_\alpha (\sqrt{-q} q^{\mu\nu}) - \nabla_\beta (\sqrt{-q} q^{\mu\beta}) \delta^\nu{}_\alpha = 0. \tag{A.7}$$

By contracting the indices α and μ, we obtain an equation which can be replaced back into (A.7), obtaining:

$$\nabla_\alpha \left(\sqrt{-q} q^{\mu\nu} \right) = 0 \quad \Rightarrow \quad \nabla_\alpha \left(q^{\mu\nu} \right) = 0. \tag{A.8}$$

This equation is a relation between the metric q and the connection Γ. In fact, this relation is the one satisfied by a metric and its Christoffel symbols. Then, Γ is the Christoffel symbol of the metric q:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} q^{\alpha\sigma} (q_{\mu\sigma,\nu} + q_{\nu\sigma,\mu} - q_{\mu\nu,\sigma}). \tag{A.9}$$

Now, we replace this into action (A.1) and get an action depending only on q, g and χ:

$$S[q, g, \chi] = -\frac{1}{2} \int d^4x \sqrt{-q} (R(q) + \frac{2}{\kappa}) - \frac{1}{\kappa} (\sqrt{-q} q^{\mu\nu} g_{\mu\nu} - 2\sqrt{-g}) + S_{\mathrm{m}}[\chi, g], \tag{A.10}$$

which is the same that (3.13).

Finally, we have concluded that (3.13), (3.10) and (A.1) are all completely equivalent.

Appendix B

Perfect Fluid

B.1 Action for a perfect fluid

A perfect fluid has the following stress-energy tensor:

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} - pg^{\mu\nu}, \tag{B.1}$$

where ρ is the energy density of the fluid in rest, p is the pressure, u^{μ} the 4-velocity and $g^{\mu\nu}$ the metric coupled to the fluid. The hydrodynamical action for a fluid with the stress-energy tensor showed above can be written as:

$$S_{\mathrm{m}} = -\int \rho\sqrt{(-g)}d^4x, \tag{B.2}$$

where g is the determinant of the metric. We are interested in the theory of linear perturbations. For that purpose, we will rewrite the energy density term ρ in the action. Let's consider the conservation equation:

$$T^{\mu\nu}{}_{;\mu} = 0. \tag{B.3}$$

Replacing here the explicit expression (B.1), we get:

$$(\rho + p)u^{\nu}{}_{;\mu}u^{\mu} + [(\rho + p)u^{\mu}]_{;\mu}u^{\nu} - p_{;\mu}g^{\mu\nu} = 0, \tag{B.4}$$

where we have used that $g^{\mu\nu}{}_{;\mu} = 0$. If we contract the equation (B.4) with u_ν, we get

$$(\rho + p)u^\nu{}_{;\mu}u^\mu u_\nu + [(\rho + p)u^\mu]_{;\mu}u^\nu u_\nu - p_{;\mu}u^\mu = 0. \tag{B.5}$$

But $u^\nu{}_{;\mu}u^\mu u_\nu = 0$ and $u^\nu u_\nu = 1$, then

$$[(\rho + p)u^\mu]_{;\mu} - p_{;\mu}u^\mu = 0 \Rightarrow (\rho + p)u^\mu{}_{;\mu} - \rho_{;\mu}u^\mu = 0. \tag{B.6}$$

Now, let's define a new variable n such that

$$\frac{dn}{n} = \frac{d\rho}{\rho + p}, \tag{B.7}$$

where we have considered that there is an equation of state $p = p(\rho)$. Then, equation (B.6) turns out

$$(\rho + p)u^\mu_{;\mu} - n_{;\mu}\frac{(\rho + p)}{n}u^\mu = 0 \quad \Rightarrow \quad (nu^\mu)_{;\mu} = 0. \tag{B.8}$$

This last equation looks like an equation of continuity of mass. Therefore, we can think that this perfect fluid is composed of many test particles with a rest mass m_0, and a number density n. Then, we can write the energy density ρ as consisting of a rest mass energy plus an energy $\Pi(n)$:

$$\rho = n[m_0 + \Pi(n)]. \tag{B.9}$$

The energy Π can be found by differentiating (B.9) and comparing it with (B.7):

$$d\rho = dn\frac{\rho}{n} + nd\Pi = dn\frac{(\rho + p)}{n} \quad \Rightarrow \quad d\Pi = dn\frac{p}{n^2}. \tag{B.10}$$

Integrating this, we obtain

$$d\Pi = -d\left(\frac{p}{n}\right) + \frac{dp}{n} \quad \Rightarrow \quad \Pi = \int \frac{dp}{n} - \frac{p}{n}. \tag{B.11}$$

Finally, we found that the hydrodynamical action for a perfect fluid is given by (B.2), where ρ is (B.9), such that (B.8) and (B.11).

It is important to write the hydrodynamical action (B.2) in terms of n because ρ is not a basic dynamical degree of freedom, then it is not possible to immediately expand the action in terms of perturbations of ρ. A basic dynamical degree of freedom is one that characterises the fluid flow, and hence it is convenient to think the fluid as formed by test particles with number density n.

B.1.1 Verification

Now, we should verify that the variation of the action $S_{\rm m}$, with respect to the metric g, gives the stress-energy tensor for a perfect fluid. This means that we should obtain:

$$\delta_g S_{\rm m} = -\frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}. \tag{B.12}$$

Let's variate the action in (B.2):

$$\delta_g S_{\rm m} = -\int\left[\frac{d\rho}{dn}\delta_g n + \rho\frac{\delta_g\sqrt{-g}}{\sqrt{-g}}\right]\sqrt{-g}. \tag{B.13}$$

But we know that

$$\frac{\delta_g\sqrt{-g}}{\sqrt{-g}} = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}. \tag{B.14}$$

Also, using (B.10), we have

$$\frac{d\rho}{dn} = \frac{\rho}{n} + n\frac{d\Pi}{dn} = \frac{(\rho+p)}{n}. \tag{B.15}$$

Replacing these two last equations in (B.13), we obtain

$$\delta_g S_{\rm m} = -\int\left[\frac{(\rho+p)}{n}\delta_g n + \frac{\rho}{2}g^{\mu\nu}\delta g_{\mu\nu}\right]\sqrt{-g}. \tag{B.16}$$

The calculation of $\delta_g n$ still remains. In order to do this, we will find n by solving equation (B.8). To solve this equation, we introduce Lagrange coordinates a^i, which label the particles in the fluid, and an affine parameter λ, which fixes the point along a particle trajectory. We write the flow in terms of the comoving coordinates as

$$x^\alpha = f^\alpha(a^i,\lambda). \tag{B.17}$$

Then, the number density ρ can be described by an arbitrary function $F(a^i)$ of the Lagrange coordinates and the Jacobean J:

$$J = \frac{\mathcal{D}(x^\alpha)}{\mathcal{D}(a^i, \lambda)}, \tag{B.18}$$

between the comoving and Lagrange coordinates. n evolves in time in the following way:

$$n(x^\mu) = \frac{F(a^i)}{\sqrt{-g}J}\sqrt{g_{\mu\nu}\frac{\partial f^\mu}{\partial\lambda}\frac{\partial f^\nu}{\partial\lambda}}. \tag{B.19}$$

It can be verified that this expression for n solves the continuity equation (B.8), if it taken into account that the cuadrivelocity is

$$u^\alpha = \frac{\frac{\partial f^\alpha}{\partial\lambda}}{\sqrt{g_{\mu\nu}\frac{\partial f^\mu}{\partial\lambda}\frac{\partial f^\nu}{\partial\lambda}}}. \tag{B.20}$$

Now, we can find $\delta_g n$ by variating (B.19). Since neither the Jacobian nor the function $F(a^i)$ depend on g, we have

$$\begin{aligned}
\delta_g\left(\rho(x^\mu)\sqrt{-g}\right) &= \frac{F(a^i)}{J}\delta_g\left(\sqrt{g_{\mu\nu}\frac{\partial f^\mu}{\partial\lambda}\frac{\partial f^\nu}{\partial\lambda}}\right)\\
&= \frac{F(a^i)}{J}\frac{g_{\mu\nu}\frac{\partial f^\mu}{\partial\lambda}\frac{\partial f^\nu}{\partial\lambda}}{\sqrt{g_{\alpha\beta}\frac{\partial f^\alpha}{\partial\lambda}\frac{\partial f^\beta}{\partial\lambda}}}\\
&= \frac{F(a^i)}{J}\frac{1}{2}\sqrt{g_{\alpha\beta}\frac{\partial f^\alpha}{\partial\lambda}\frac{\partial f^\beta}{\partial\lambda}}u^\mu u^\nu\delta g_{\mu\nu} \quad \text{because of (B.20)}\\
&= \frac{1}{2}\rho\sqrt{-g}u^\mu u^\nu\delta g_{\mu\nu} \quad \text{because of (B.19)}
\end{aligned} \tag{B.21}$$

Then, using this las result along with (B.14), we find that

$$\delta_g n = \frac{n}{2}\left[u^\mu u^\nu - g^{\mu\nu}\right]\delta g_{\mu\nu}. \tag{B.22}$$

Replacing this into (B.13), we finally get:

$$\delta_g S_{\rm m} = -\frac{1}{2}\int\left[(\rho+p)u^\mu u^\nu - pg^{\mu\nu}\right]\delta g_{\mu\nu}\sqrt{-g}. \tag{B.23}$$

By comparing this with (B.12), it follows that the stress-energy tensor is exactly that of a perfect fluid.

B.2 Second order action for a perfect fluid

Let's consider now perturbations up to second order in the action.

We saw that the hydrodynamical action is

$$S_{\rm m} = -\int \rho\sqrt{(}-g), \tag{B.24}$$

where

$$\rho = n[m_o + \Pi], \quad \text{with} \quad \Pi = \int \frac{dp}{n} - \frac{p}{n}. \tag{B.25}$$

We can expand this action up to second order assuming a background solution and variations of the metric and n:

$$S_{\rm m}^{(2)} = -\int \rho_0\sqrt{-g}^{(2)} + (\rho_0+p_0)\left(\frac{n^{(1)}}{n_0}\sqrt{-g}^{(1)} + \frac{n^{(2)}}{n_0}\sqrt{-g}^{(0)}\right) + \frac{\omega}{2}(\rho_0+p_0)\frac{(n^{(1)})^2}{n_0^2}\sqrt{-g}^{(0)}, \tag{B.26}$$

where $\sqrt{-g}$ and n were expanded up to second order as: $n = n_0 + n^{(1)} + n^{(2)}$ and $\sqrt{-g} = \sqrt{-g}^{(0)} + \sqrt{-g}^{(1)} + \sqrt{-g}^{(2)}$. Also, here we have used that $p = \omega\rho$.

Variations of $\sqrt{-g}$ are only due to metric perturbations, but variations in n are due to metric perturbations and matter perturbations. In order to calculate variations of $\sqrt{-g}$ we must use the perturbed metric and then perform a taylor expansion up to order 2 in all the perturbation variables. On the other hand, to calculate variations of $n(x)$ we must first express matter perturbations with a vector χ^μ, which shifts the flow of the fluid. Then, the full flow is

$$x^\alpha = f^\alpha(a^i,\lambda) = f_0^\alpha(a^i,\lambda) + \chi^\alpha, \tag{B.27}$$

where $f_0^\alpha(a^i,\lambda)$ is the flow in the background universe.

Second, we express $n(x)$ in terms of $n(x+\chi)$ and expand it up the second order in χ. We evaluate $n(x+\chi)$ using (B.19) and calculating separately the individual terms in this formula up to second order in all the perturbation variables (χ and metric perturbations).

B.2.1 Particular case

If we consider the particular case of a perturbed metric given by (3.37), some individual terms of (B.19) are

$$J(x+\chi) = 1 + \frac{\partial \chi^\alpha}{\partial x^\alpha} + \frac{1}{2}\left(\frac{\partial \chi^\alpha}{\partial x^\alpha}\right)^2 - \frac{1}{2}\frac{\partial \chi^\alpha}{\partial x^\beta}\frac{\partial \chi^\beta}{\partial x^\alpha}, \tag{B.28}$$

$$\begin{aligned}\sqrt{-g}(x+\chi) =& \sqrt{-g}(x)\left\{1 + 4\mathcal{H}\chi^0 + 2\left(\mathcal{H}' + 4\mathcal{H}^2\right)(\chi^0)^2\right.\\ &\left. + (\phi_2 - 3\psi_2 + E_{2,ii})'\chi^0 + (\phi_2 - 3\psi_2 + E_{2,ii})_{,j}\chi^j\right\},\end{aligned} \tag{B.29}$$

where $\mathcal{H} = a'/a$, and $\sqrt{-g}(x)$ is given by (3.71). We can continue doing this, and we will obtain

$$\begin{aligned}n = n_0 + n^{(1)} + n^{(2)} =& n_0\Bigg[1 + \left(3\psi_2 - E_{2,ii} - \chi^i{}_{,i}\right) + \Bigg(-B_{2,i}\chi^{i'} - \frac{1}{2}\chi^{i'}\chi^{i'}\\ & - \frac{1}{2}B_{2,i}B_{2,i} + \frac{15}{2}\psi_2^2 + \frac{1}{2}E_{2,ii}E_{2,jj} + E_{2,ij}E_{2,ji} - 5\psi_2 E_{2,ii}\\ & -(3\psi_2 - E_{2,ii})\chi^i_{,i} + (\chi^0\chi^{i'} + \frac{1}{2}\chi^i{}_{,j}\chi^j + \frac{1}{2}\chi^j{}_{,j}\chi^i)_{,i}\Bigg)\Bigg],\end{aligned} \tag{B.30}$$

where $\chi^i = \chi^{,i}$. Replacing this result into (B.26), we obtain the action (3.76), up some total derivatives.

For vector and tensor perturbations, calculations are completely analogous. If we had considered (3.40), we would have obtained:

$$n = n_0[1 - S_{2i}\xi^{i'} - \frac{1}{2}\chi^{i'}\chi^{i'} - \frac{1}{2}S_{2i}S_{2i} + (F_{2i,j} + F_{2j,i})(F_{2i,j} + F_{2j,i}) + \frac{1}{2}\chi^i{}_{,j}\chi^j{}_{,i}], \tag{B.31}$$

where $\xi^i{}_{,i} = 0$.

If we had considered (3.49), we would have obtained:

$$n = n_0[1 + \frac{1}{4}h_{ij}h_{ji}]. \tag{B.32}$$

B.3 Perturbed stress-energy tensor

Also, in this particular case, we can calculate the perturbed stress-energy tensor up to first order. We split the tensor as:

$$T^{\mu}{}_{\nu} = {}^{(0)}T^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}, \tag{B.33}$$

where ${}^{(0)}T^{\mu}{}_{\nu}$ is the background stress-tensor, and $\delta T^{\mu}{}_{\nu}$ is given by

$$\delta T^{0}{}_{0} = \rho^{(1)} = \frac{(\rho_0 + p_0)}{n_0} n^{(1)} = (\rho_0 + p_0)(3\psi_2 - E_{2,ii} - \chi^{i}_{,i}), \tag{B.34}$$

$$\delta T^{i}{}_{0} = (\rho_0 + p_0) u^{(1)i} u^{(0)}_{0} = (\rho_0 + p_0)\chi^{i'}, \tag{B.35}$$

$$\delta T^{0}{}_{i} = (\rho_0 + p_0) u^{(0)0} u^{(1)}_{i} = -(\rho_0 + p_0)(B_{2,i} + \chi^{i'} - S_{2i}), \tag{B.36}$$

$$\delta T^{i}{}_{j} = -p^{(1)}\delta^{i}_{j} = -\omega\rho^{(1)}\delta^{i}_{j} = -\omega(\rho_0 + p_0)(3\psi_2 - E_{2,ii} - \chi^{i}_{,i})\delta^{i}_{j}, \tag{B.37}$$

where we have calculated $u^{(1)\mu}$ using (B.20), obtaining

$$u^{(0)0} = -\frac{\phi_2}{a} - \frac{\mathcal{H}}{a}\chi^{0}, \quad u^{(0)i} = \frac{\chi^{i'}}{a}. \tag{B.38}$$

This stress-energy tensor $\delta T^{\mu}{}_{\nu}$ is exactly the same to that showed in equations (3.38) and (3.41).

Bibliography

Allen, B., “Vacuum states in de Sitter space”, Phys. Rev. D 32, 3136-3149 (1985)

Armendariz-Picon, C., “Why should primordial perturbations be in a vacuum state”, JCAP 0702:031, 2007.

Avelino, P., & Ferreira, R., “Bouncing Eddington-inspired Born-Infeld cosmologies: an alternative to Inflation?”, Phys. Rev. D 86, 041501 (2012).

Baccigalupi, C., “Linear cosmological perturbations and cosmic microwave background anisotropies”, Lecture notes for a course at SISSA, Italy, 2012.

Bañados, M. & Ferreira, P.,“Eddington’s theory of gravity and its progeny”, Phys. Rev. Lett. 105, 011101 (2010).

Baumann, D., “On the quantum origin of structure in the inflationary universe”, Lecture notes for Princeton University, 2007 .

Born, M. & Infeld, L., “Foundations of the new field theory”, Proc. Roy. Soc. Lond, A144, 425-451 (1934).

Dodelson, S., “Modern Cosmology”, Academic Press (2003).

Eddington, A., “The Mathematical Theory of Relativity”, Cambridge University Press (1924).

Escamilla-Rivera, C. & Bañados, M. & Ferreira, P., “A tensor instability in the Eddington inspired Born-Infeld Theory of Gravity”, Phys. Rev. D 85, 087302 (2012).

Fock, V., “The theory of spacetime and gravitation”, Macmillan (1954).

Garcia-Bellido, J., “Astrophysics and Cosmology”, Lectures at European School of High Energy Physics, Casta-Papiernicka, Slovak Republic, 1999.

Guth, A., “Inflationary universe: a possible solution to the horizon and flatness problem”, Phys. Rev. D 23, 347-356 (1981).

Harrison, E., “Fluctuations at the Threshold of Classical Cosmology”, Phys. Rev. D 1, 2726-2730 (1970).

Helbig, P., “Is there a flatness problem in classical cosmology?”, MNRAS, 421, 1, 561–569 (2012).

Hollands, S. & Wald, R., “Alternative to Inflation”, Gen. Rel. Grav. 34, 2043 (2002).

Komatsu, E., et al., “Seven-year Wilkinson microwave anisotropy probe (WMAP1) observations: cosmological interpretation”, The Astrophysical Journal Supplement, Volume 192, Issue 2, article id. 18, 47 pp. (2011).

Maldacena, J., “Non-Gaussian features of primordial fluctuations in single field inflationary models”,JHEP 0305 (2003) 013.

Mukanov, V., “Physical foundations of cosmology”, Cambridge University Press (2005).

Mukanov, V. & Feldman, H. & Brandenberger, R., “Theory of cosmological perturbations”, Phys. Reports 215, Nos. 5 and 6 (1992) 203-333.

Mukanov, V. & Winitzki, S., “Introduction to quantum fields in classical backgrounds”, Lecture notes, draft version 2004.

Prokopec, T., “Lecture notes on Cosmology”, Lectures for a cosmology course at Utrecht University, 2012.

Riotto, A., “Inflation and the theory of cosmological perturbations”, Lectures delivered at the ”ICTP Summer School on Astroparticle Physics and Cosmology”, Trieste, Italy, 2002.

Vollick, D., “Palatini approach to Born-Infeld-Einstein theory and a geometric description of electrodynamics”, Phys. Rev. D 69, 064030 (2004).